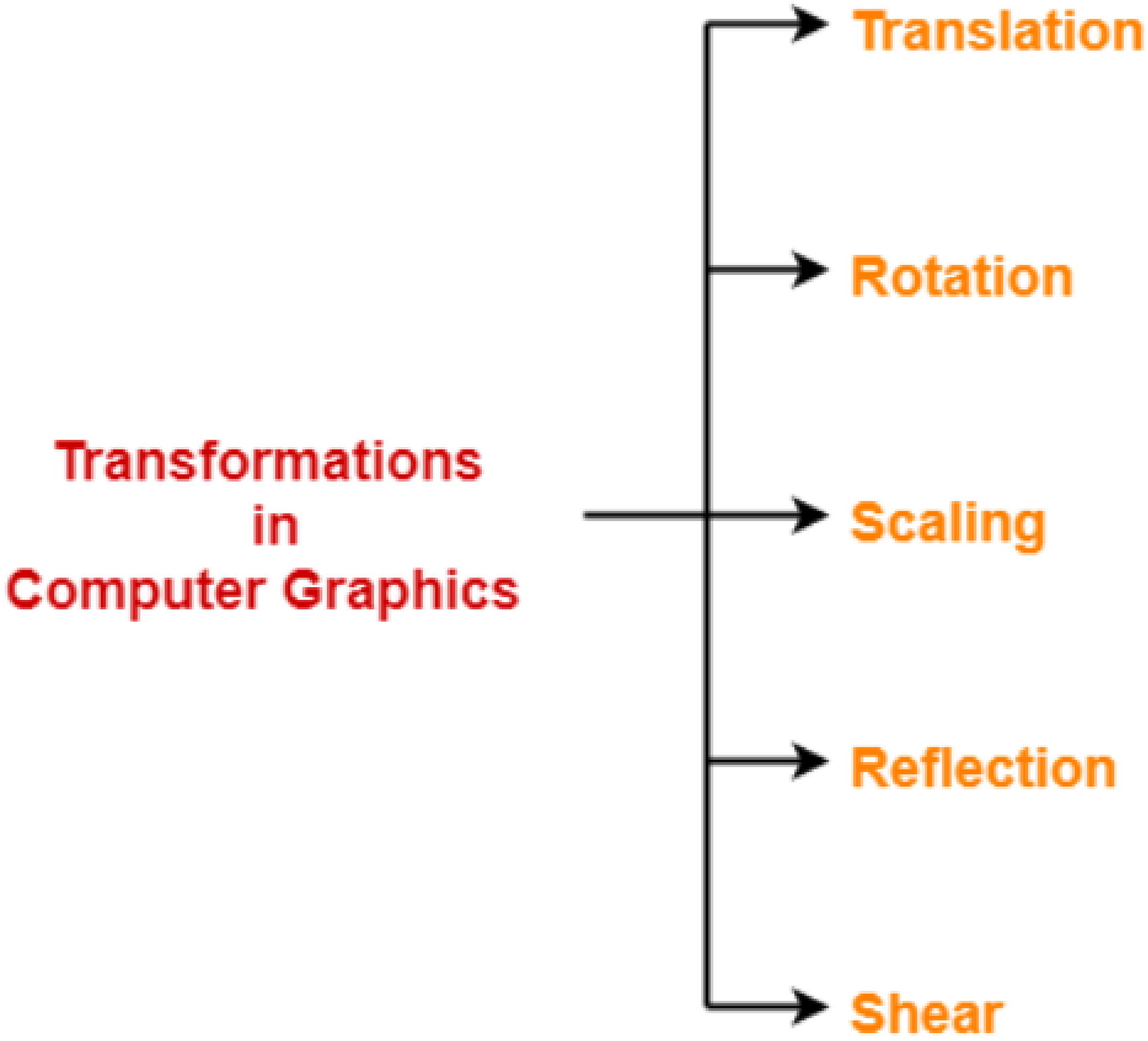
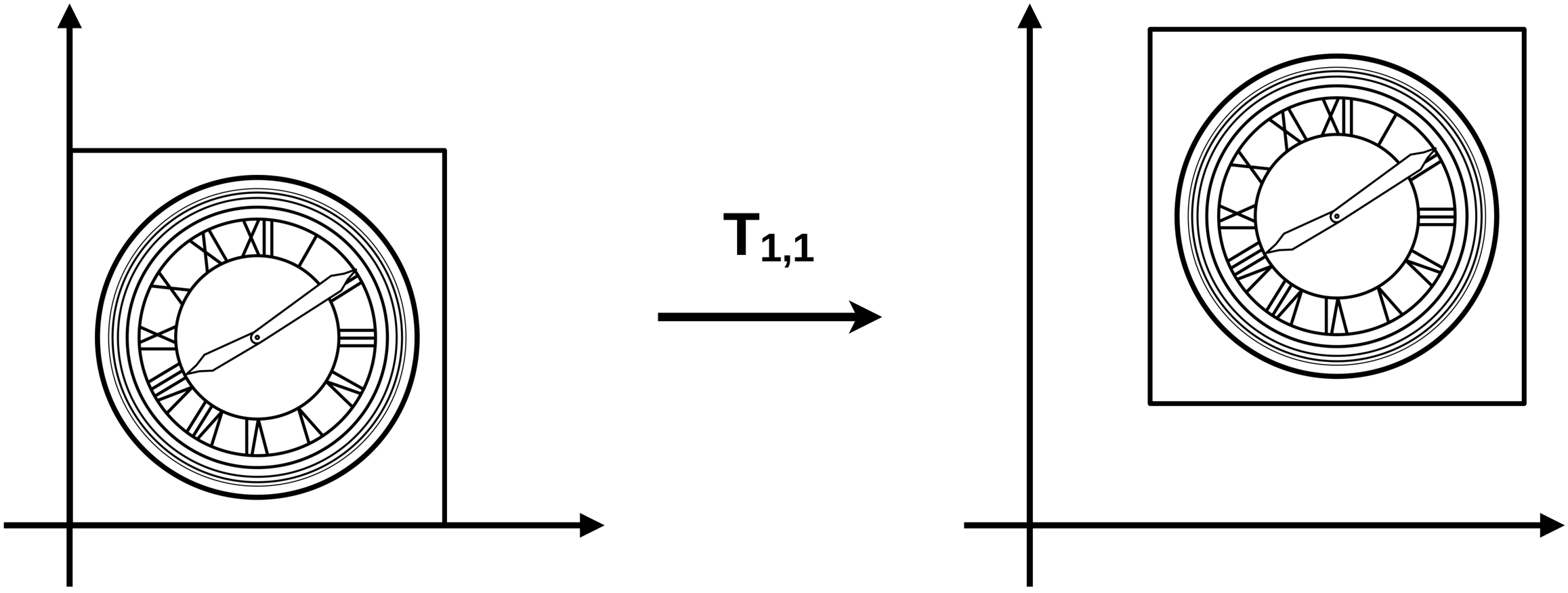
Lecture 2:

Transforms 2D

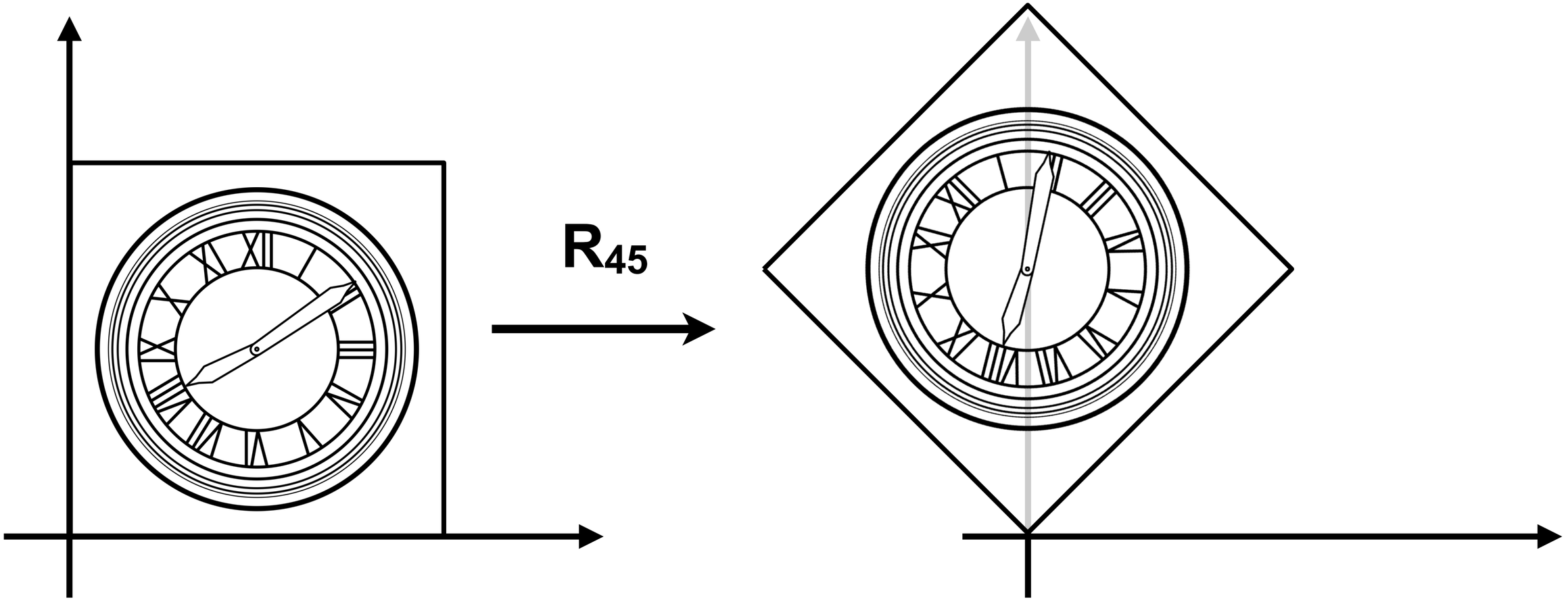
Computer Graphics and Imaging


Transformations
in
Computer Graphics
Translation
Rotation
Scaling
Reflection
Shear

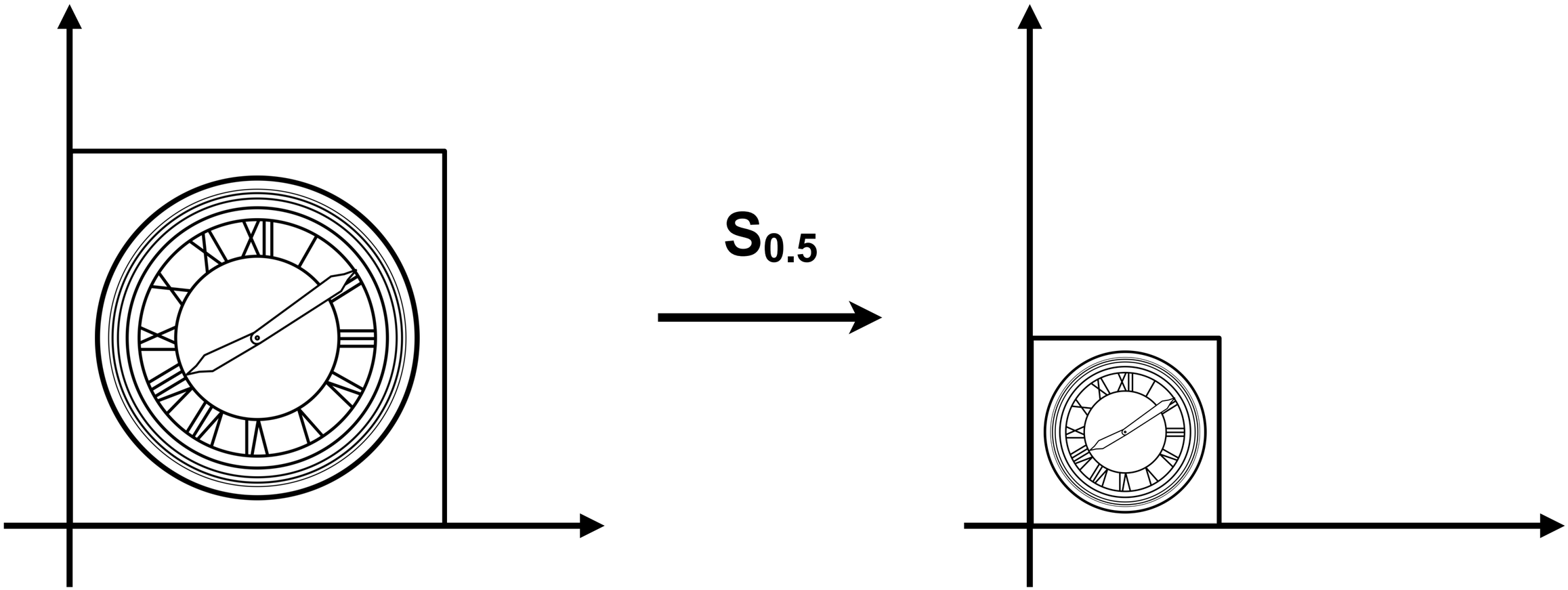
Translate



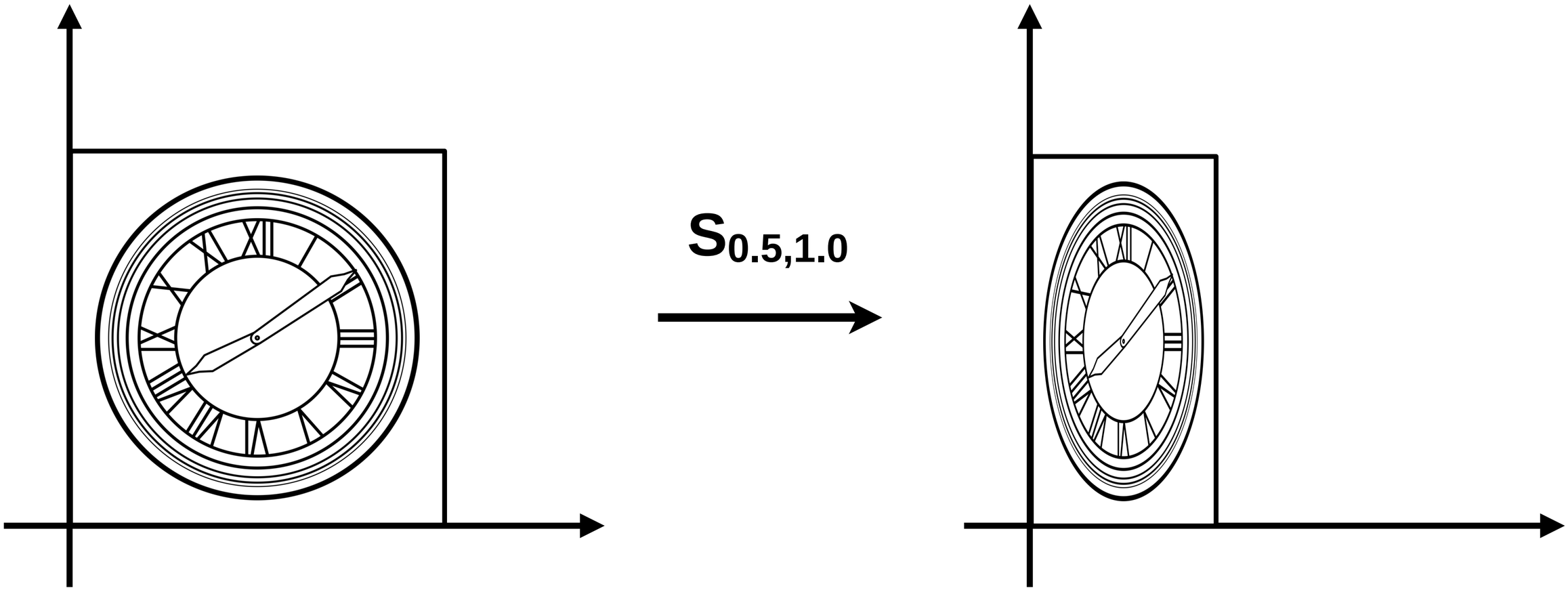
Rotate



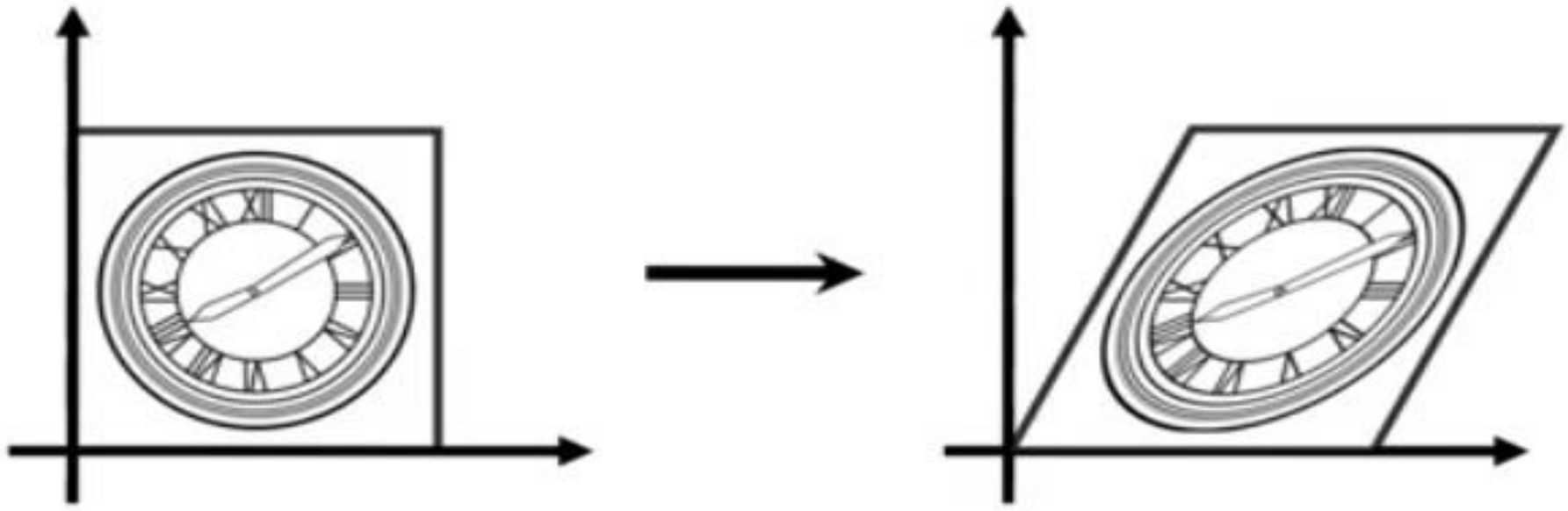
Scale



Scale (Non-Uniform)



Shear Matrix

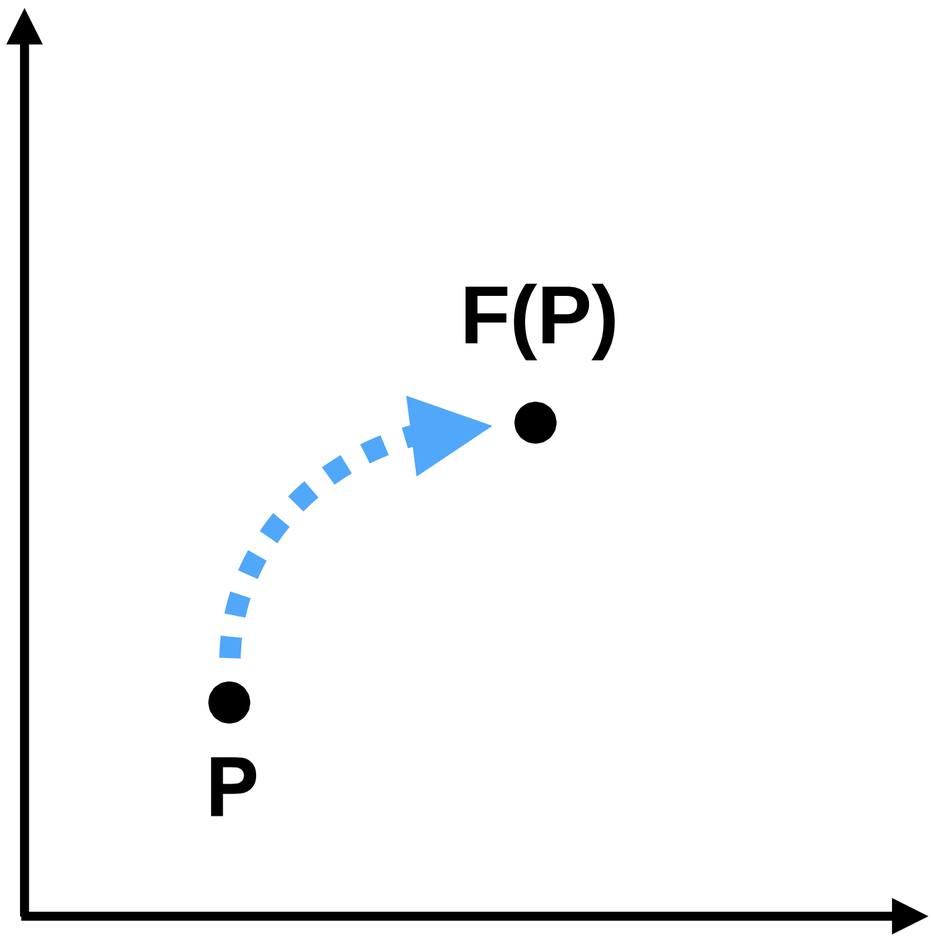


$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

What Are Transforms?

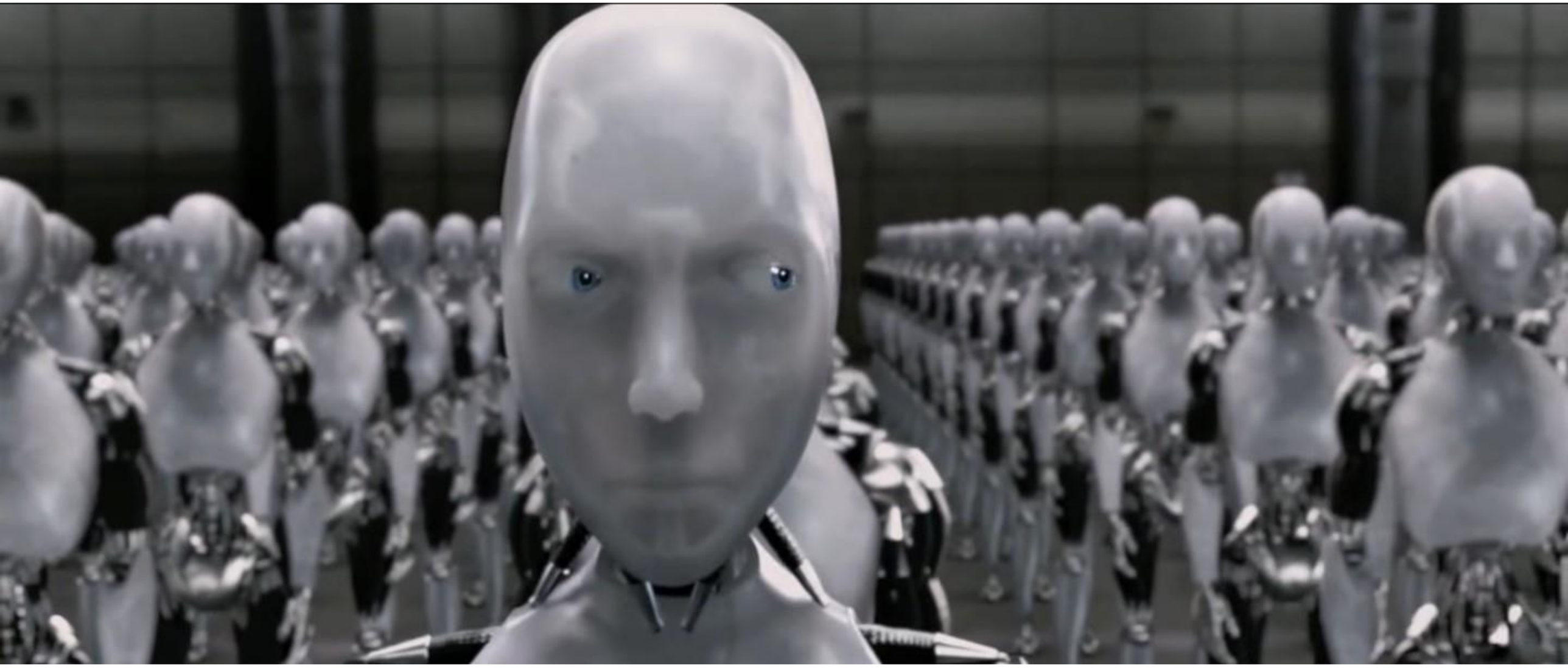
Just functions acting on points

- **(x’,y’,z’) = F(x,y,z)**
- **P’ = F(P)**



Why Study Transforms?

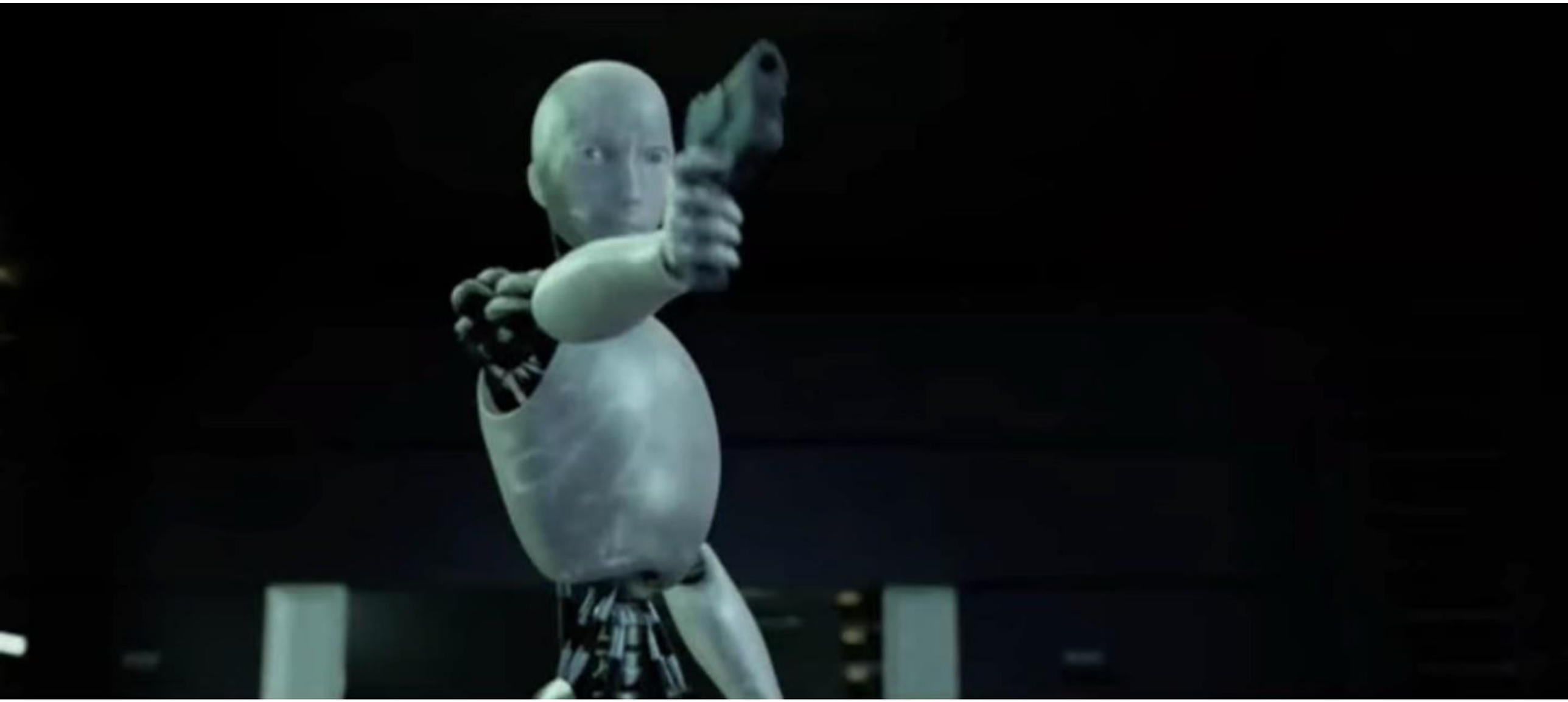
Modeling A Robot Army



iRobot movie

Transforms can describe position of object instances

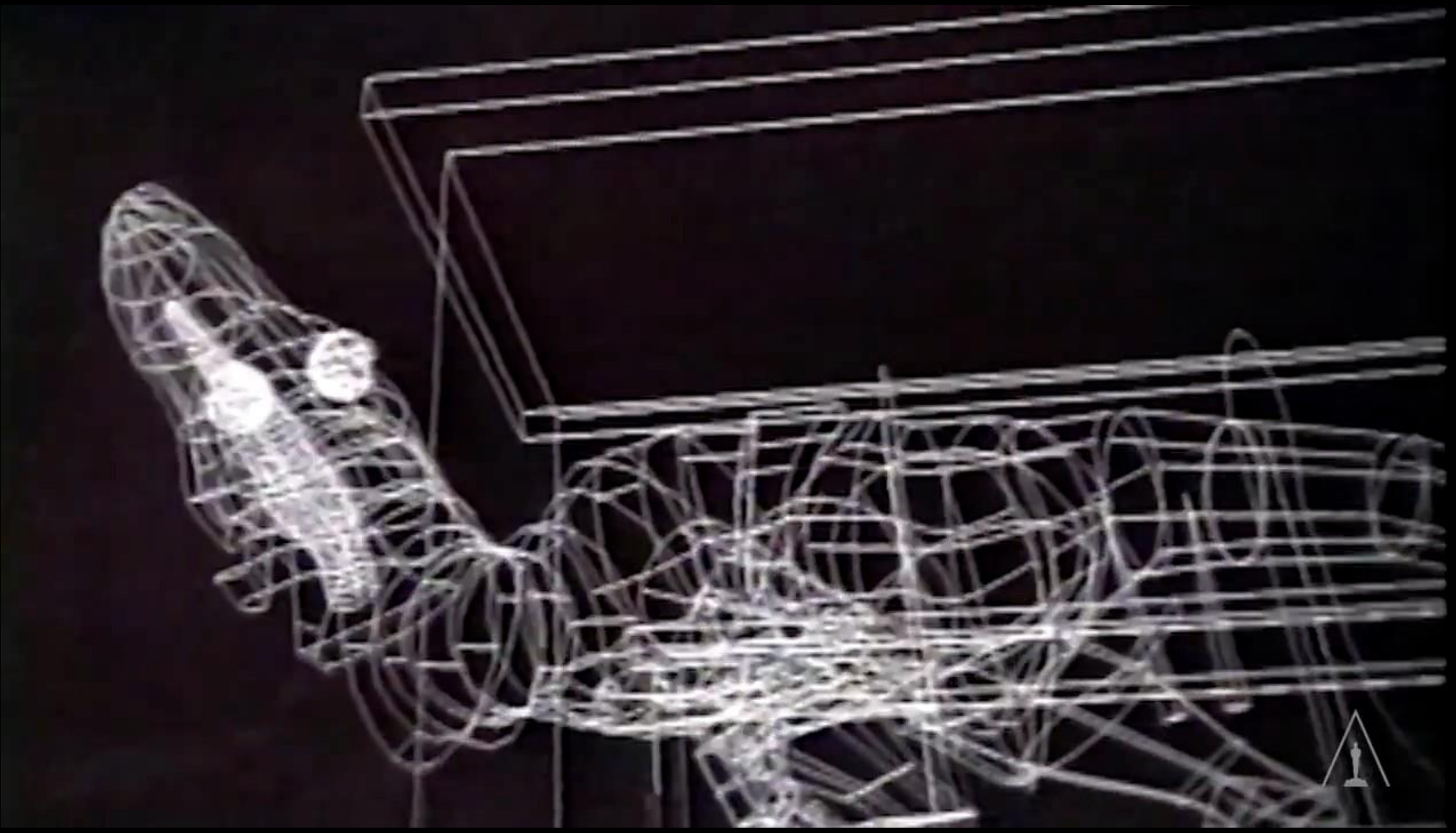
Posing a Character’s Skeleton



iRobot movie

Transforms can describe relative position of connected body parts

Project Polygons in 3D to 2D Screen

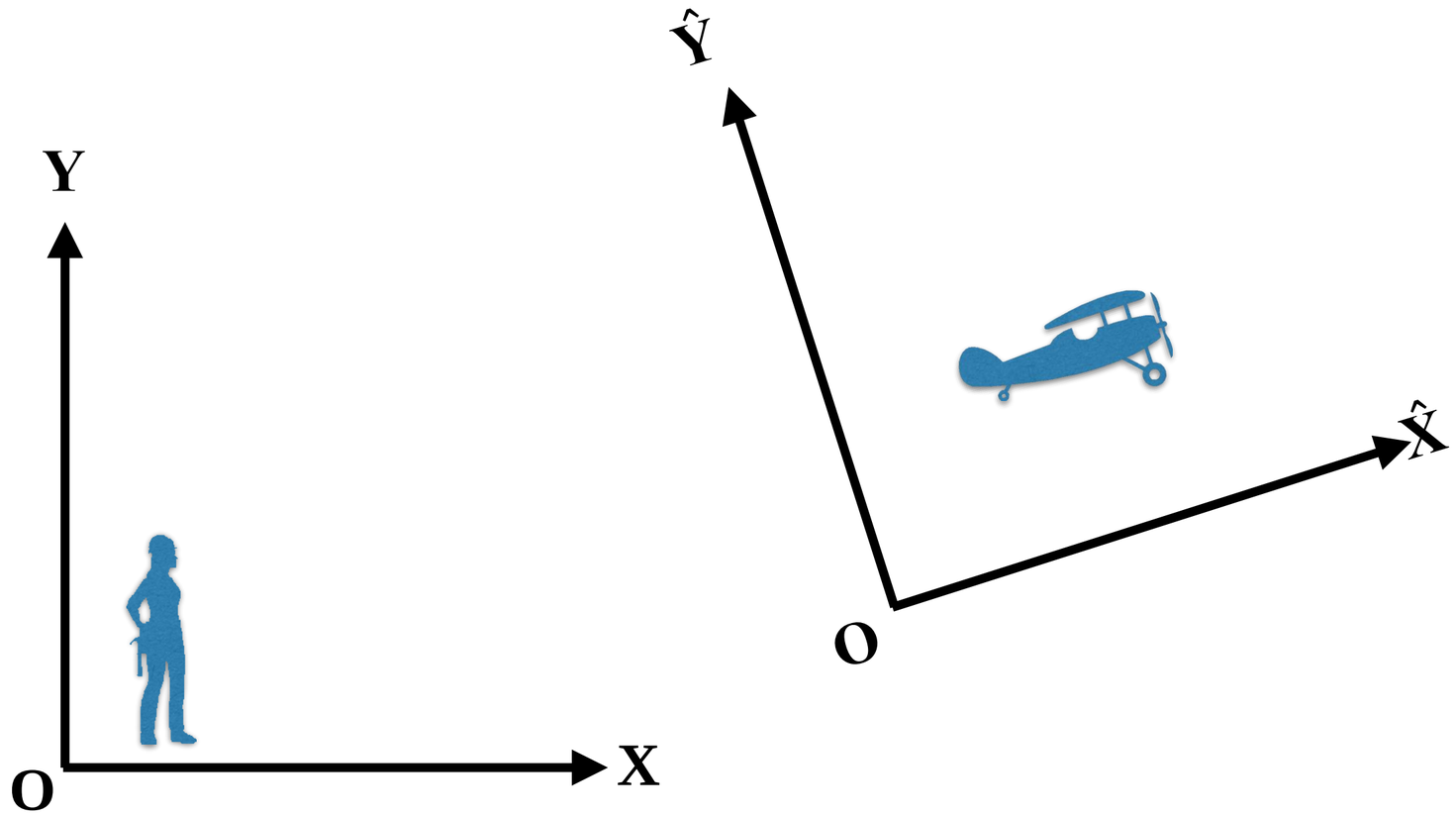


Moments That Changed The Movies: Jurassic Park

https://www.youtube.com/watch?v=KWsbcBvYqN8

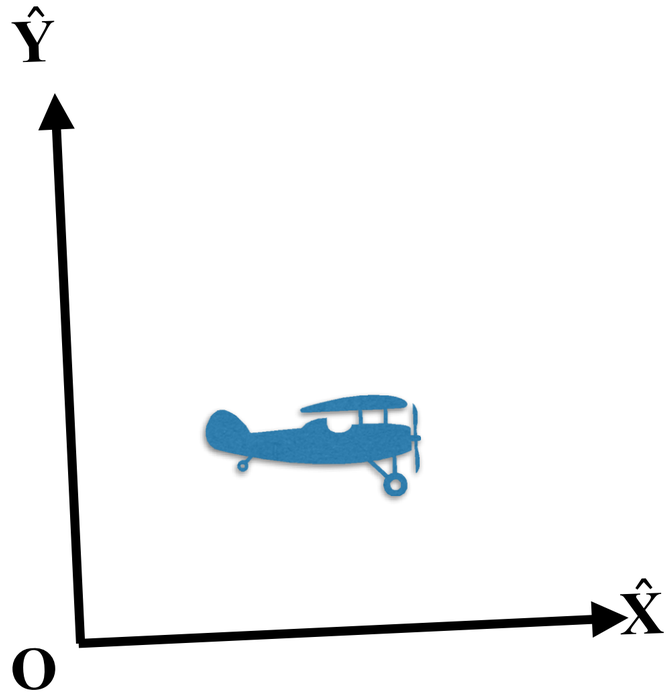
Coordinate Systems are Arbitrary

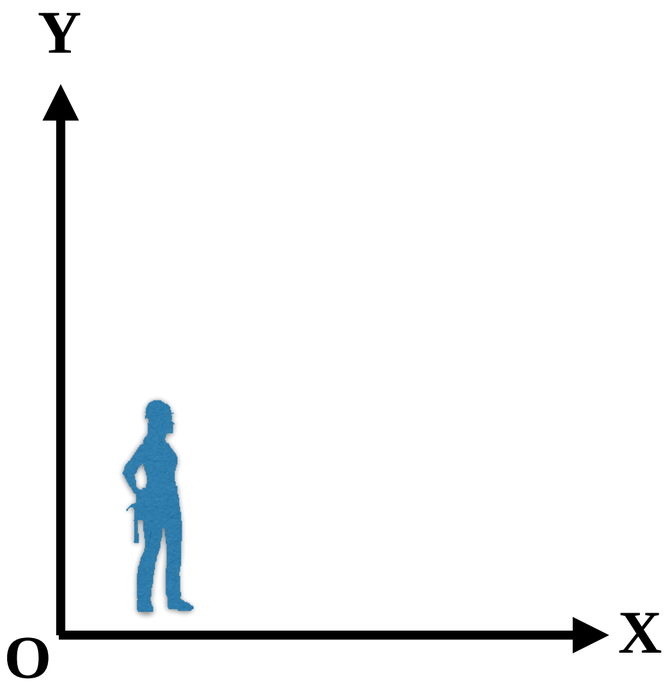
So feel free to pick one you like!



Coordinate Systems are Arbitrary

So feel free to pick one you like!





Why Study Transforms?

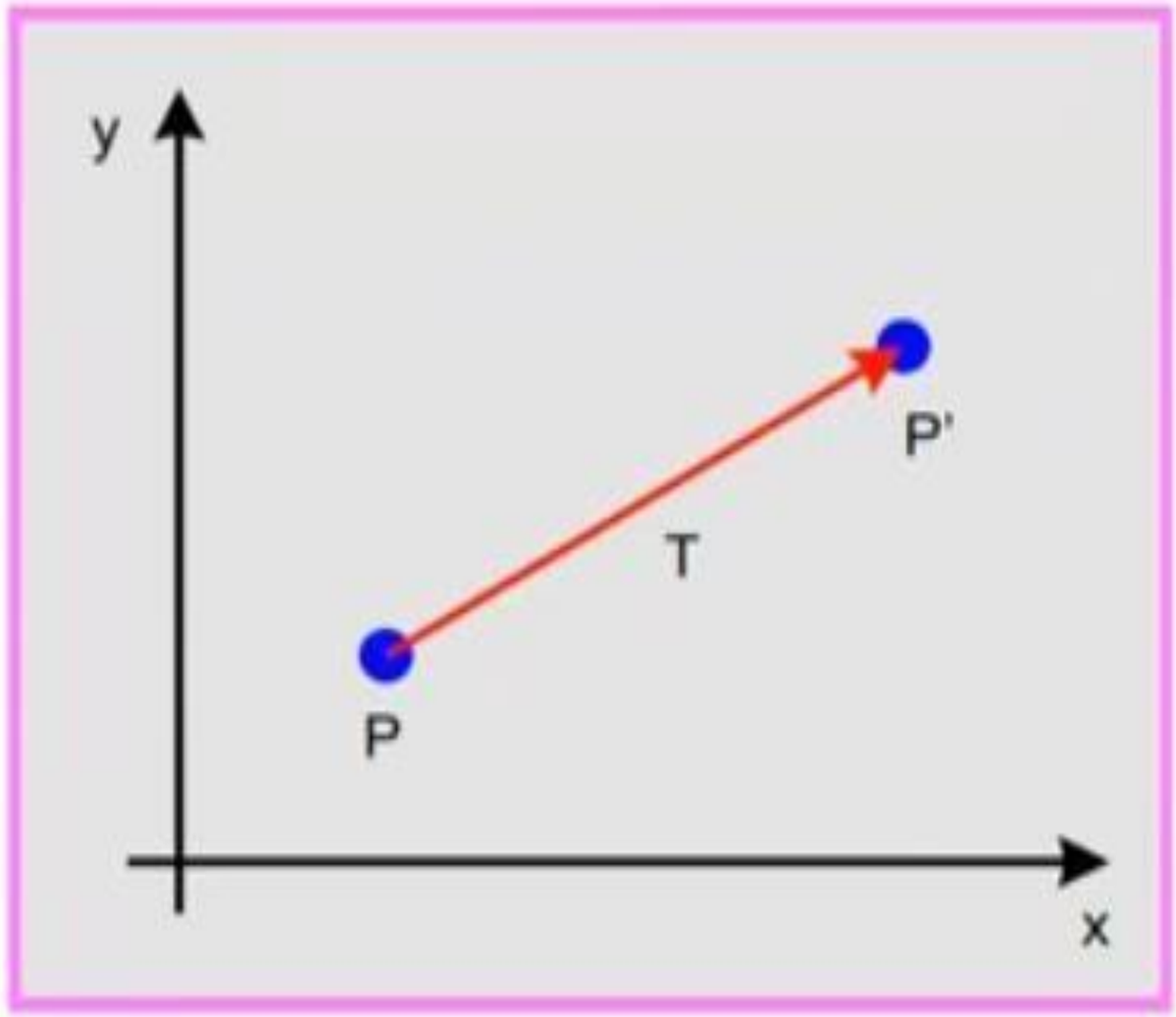
Modeling

- **Define shapes in convenient coordinates**
- **Enable multiple copies of the same object**
- **Efficiently represent hierarchical scenes Viewing**
  - **World coordinates to camera coordinates**
- **Parallel / perspective projections from 3D to 2D Other**
  - **Switching between coordinate systems**

Linear Transforms = Matrices

Translation

- **2D Translation:** Move a point along a straight-line path to its new location.

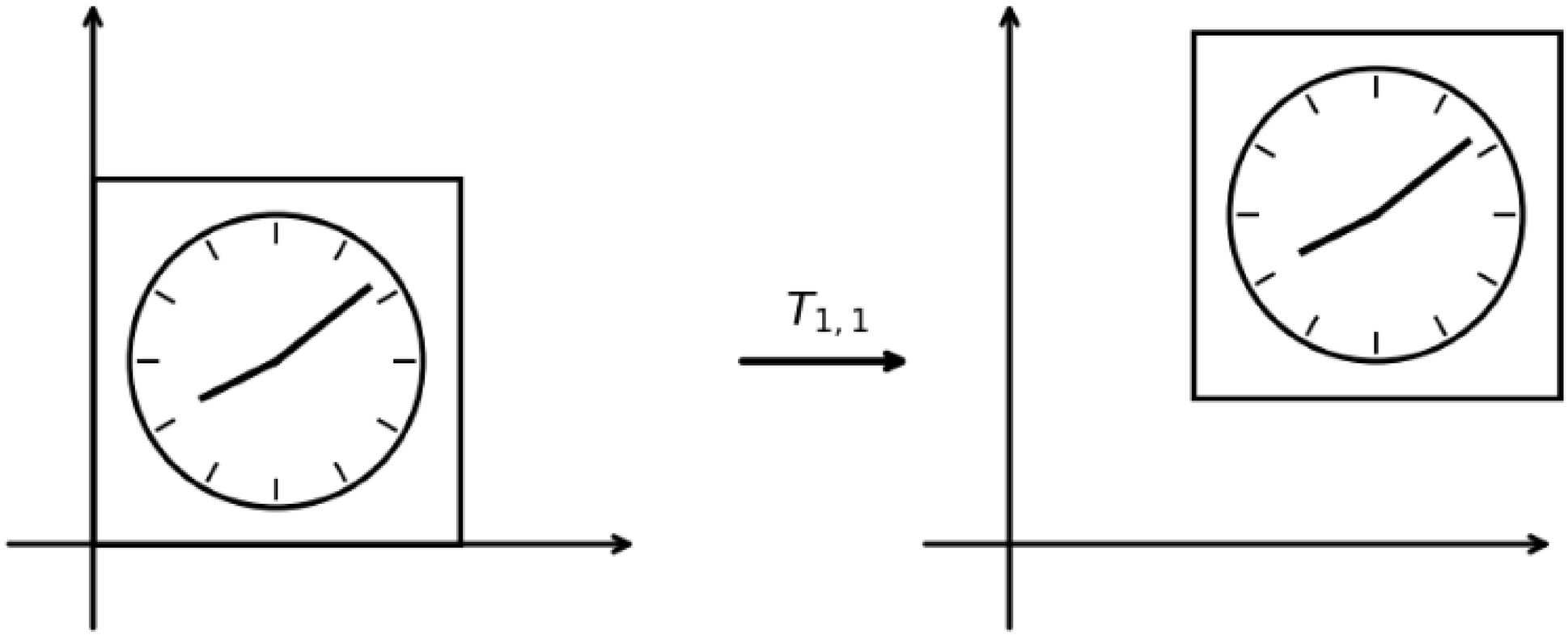


$$x' = x + t_x, \qquad y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$P' = P + T$$

Translation



$$x' = x + t_x$$

$$y' = y + t_y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 & tx \\ 0 & 1 & ty \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Translation

Representation

1. In-homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\dot{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\dot{\mathbf{p}}_1} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\mathbf{t}}$$

2. Homogeneous coordinates

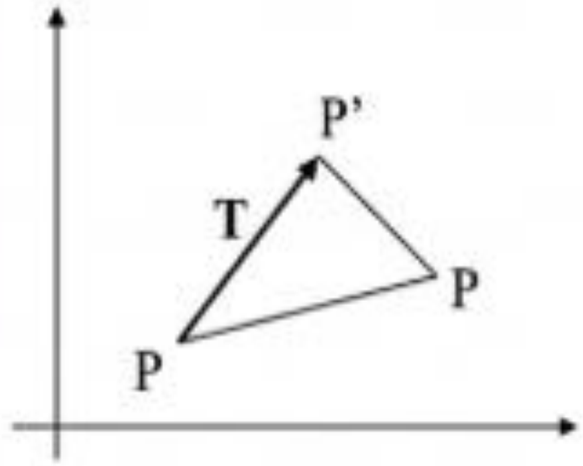
$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}$$

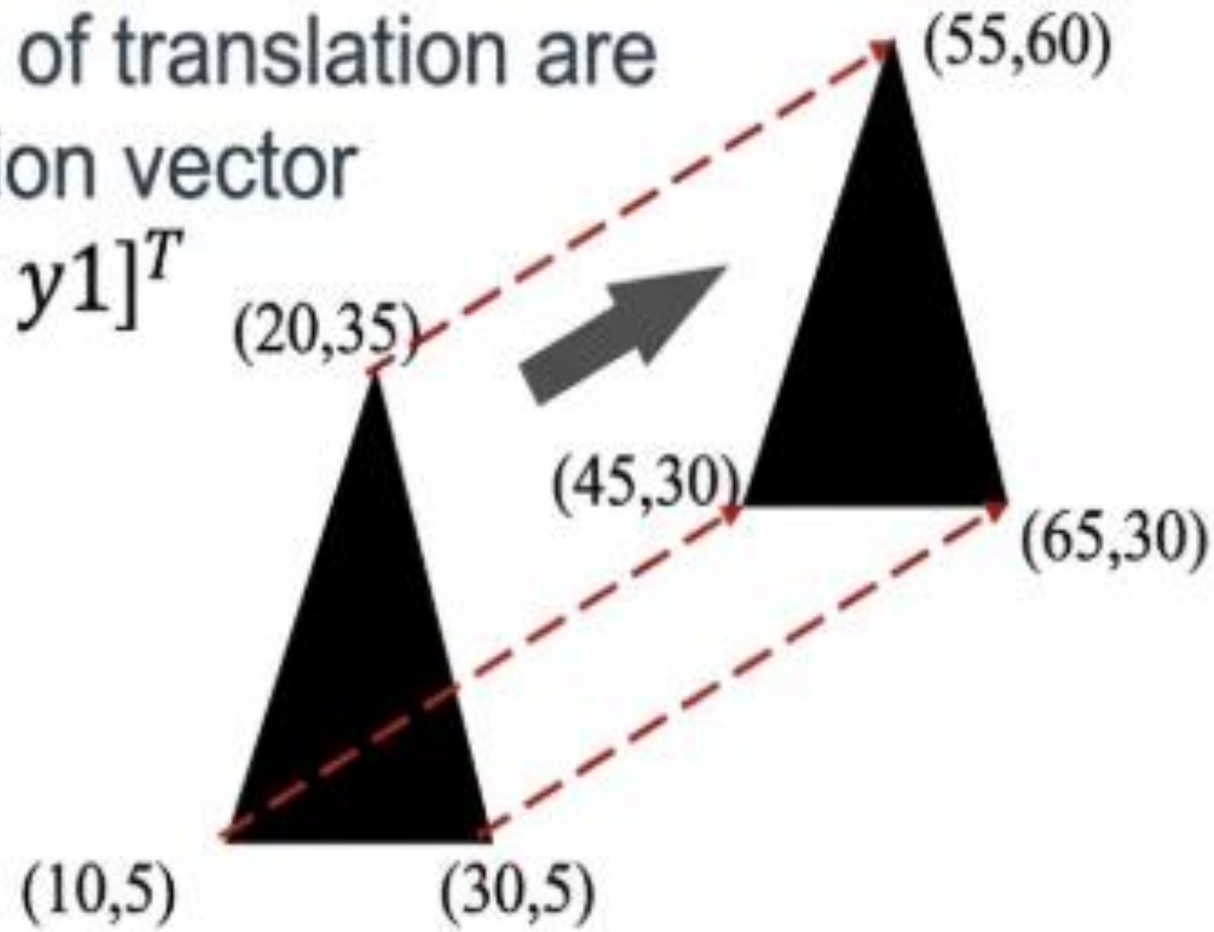
2D Translation

glTranslated(GLdouble x, GLdouble y, GLdouble z)

- Translation in 2D space is performed when a 2D point is moved from $[x1, y1]^T$ to another point $[x2, y2]^T$.
- The magnitude and direction of translation are characterized by the translation vector $\boldsymbol{t} = [tx, ty]^T = [x2 - x1, y2 - y1]^T$

$$x' = x + t_x$$
$$y' = y + t_y$$





Example

Consider a triangle with vertices located at **a1** = $[1,0]^T$, **b1** = $[0,1]^T$ and **c1** = $[-1,0]^T$. This triangle is translated using a translation vector **t** = $[2,3]^T$. Determine the new locations of its vertices a2, b2 and c2.

Solution

$$\mathbf{a}_2 = \underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{\mathbf{a}_1} + \underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix}3\\3\end{bmatrix}}_{\mathbf{a}_2}$$

$$\mathbf{b}_2 = \underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{\mathbf{c}_1} + \underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix}2\\4\end{bmatrix}}_{\mathbf{b}_2}$$

$$\mathbf{c}_2 = \underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{\mathbf{c}_1} + \underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{\mathbf{t}} = \underbrace{\begin{bmatrix}1\\3\end{bmatrix}}_{\mathbf{c}_2}$$

OR

$$\mathbf{a}_2 = \underbrace{\begin{bmatrix}1&0&2\\0&1&3\\0&0&1\end{bmatrix}}_{\tau([2,3]^T)} \underbrace{\begin{bmatrix}1\\0\\1\end{bmatrix}}_{\mathbf{a}_1} = \underbrace{\begin{bmatrix}3\\3\\1\end{bmatrix}}_{\mathbf{a}_2}$$

$$\mathbf{b}_2 = \underbrace{\begin{bmatrix}1&0&2\\0&1&3\\0&0&1\end{bmatrix}}_{\tau([2,3]^T)} \underbrace{\begin{bmatrix}0\\1\\1\end{bmatrix}}_{\mathbf{a}_1} = \underbrace{\begin{bmatrix}2\\4\\1\end{bmatrix}}_{\mathbf{b}_2}$$

$$\mathbf{c}_2 = \underbrace{\begin{bmatrix}1&0&2\\0&1&3\\0&0&1\end{bmatrix}}_{\tau([2,3]^T)} \underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\mathbf{b}_1} = \underbrace{\begin{bmatrix}1\\3\\1\end{bmatrix}}_{\mathbf{b}_2}$$

Example

Translate a polygon with coordinates A(2,7), b(7,10), c(10,2) by 3 unites in x direction and 4 units in Y direction

Solution

A’= (5,11)

B’=(10,14)

C’=(13,6)

example

$$A' = A + T$$

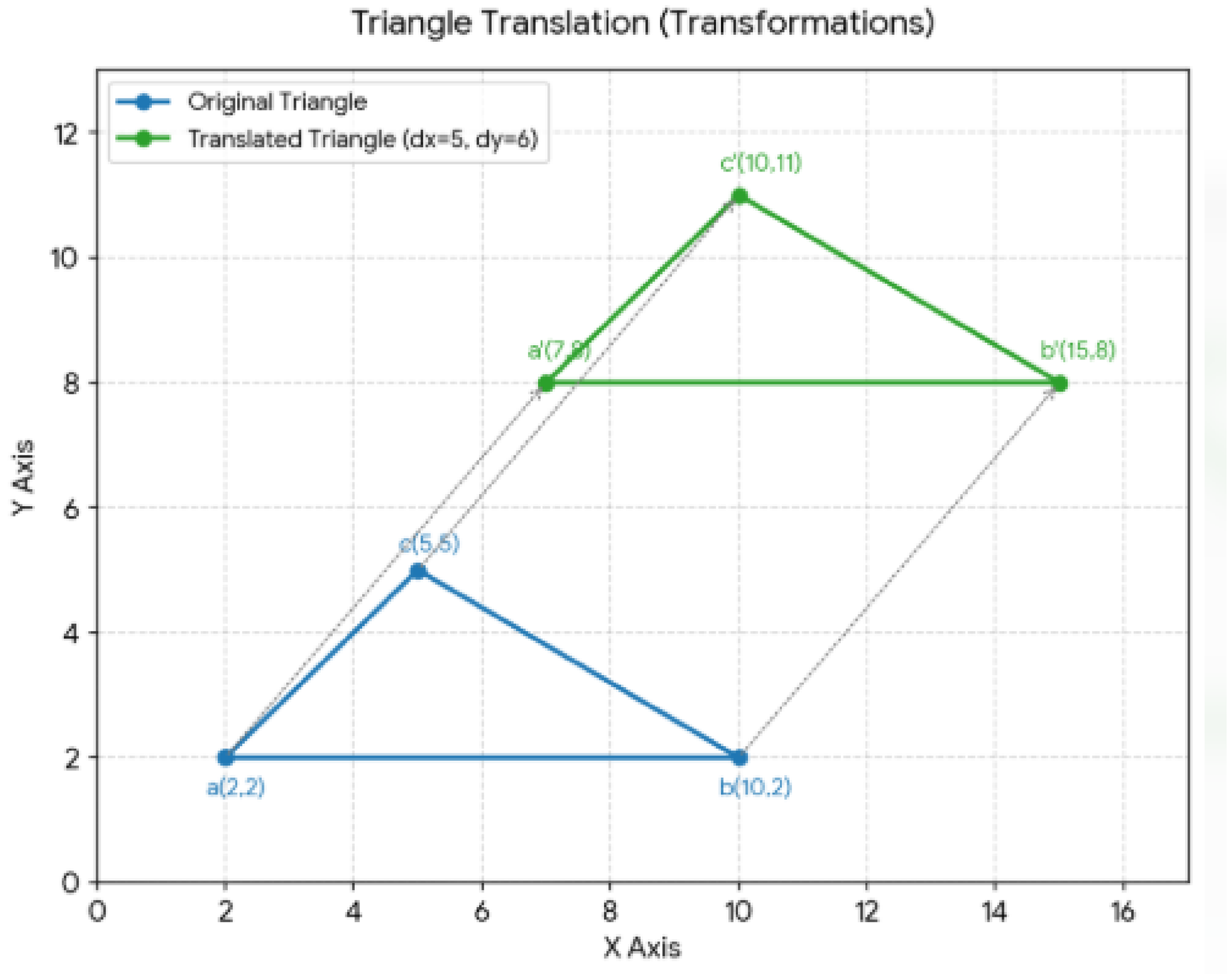
$$= \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$B' = B + T$$

$$= \begin{bmatrix} 10 \\ 2 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 15 \\ 8 \end{bmatrix}$$

$$c' = C + T$$

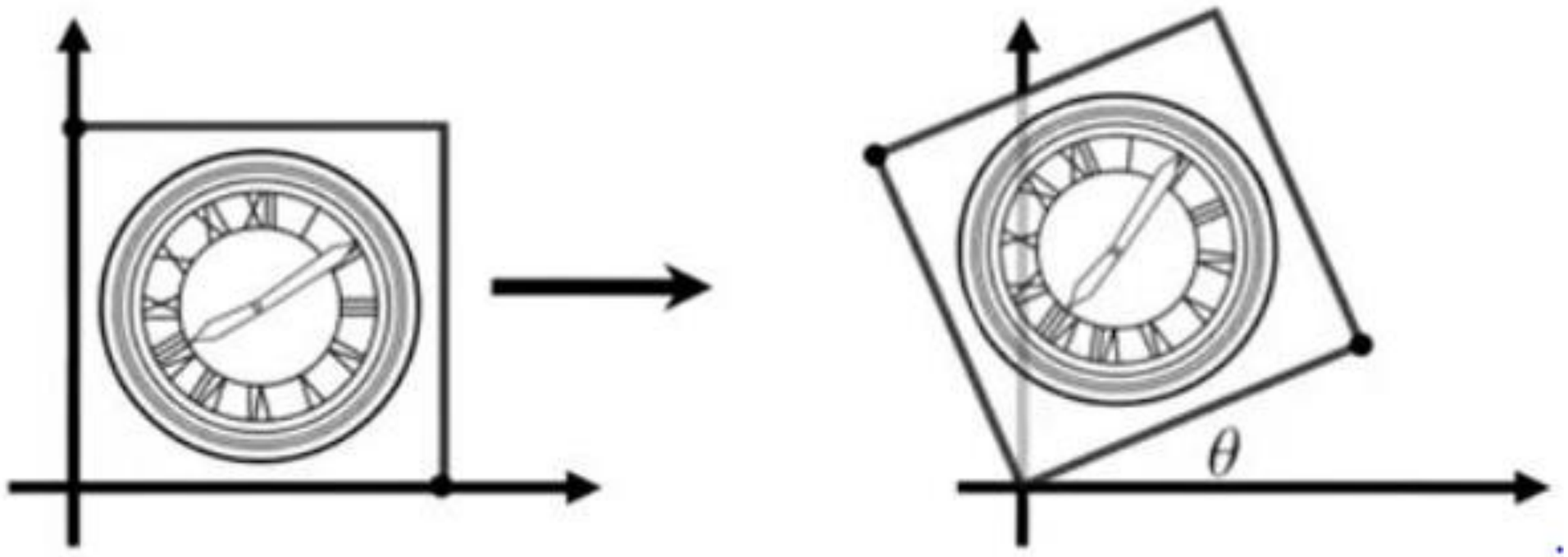
$$= \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$



2D Rotation

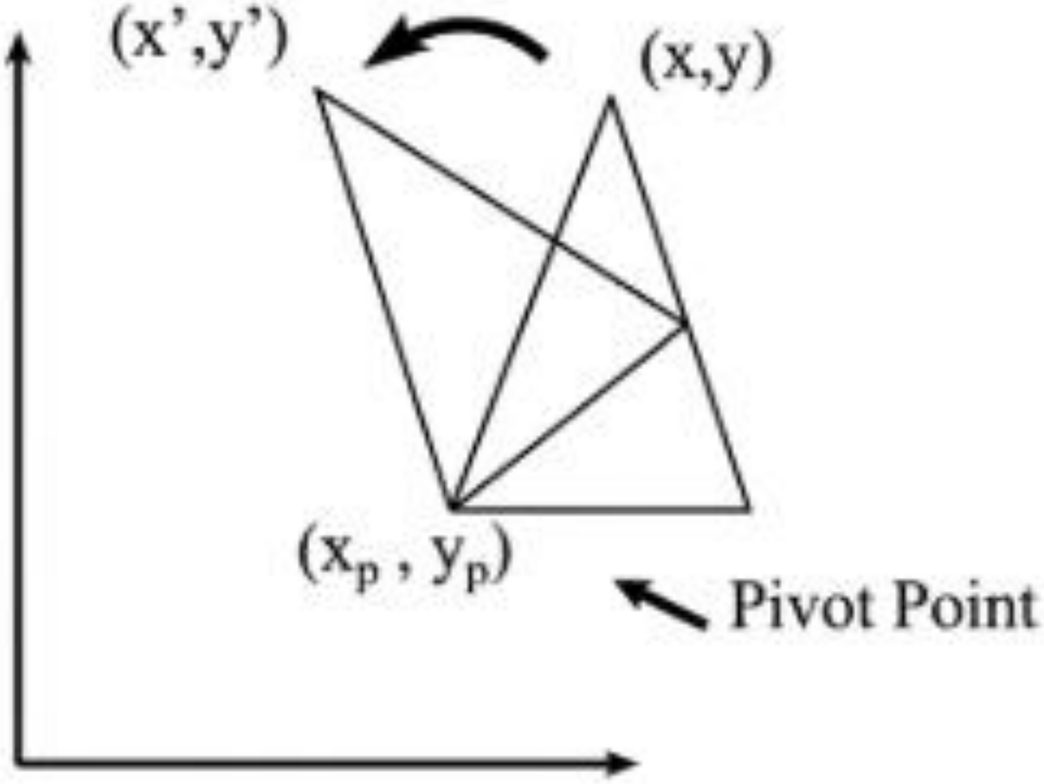
glRotate(GLdouble angle, GLdouble x, GLdouble y, GLdouble z)

- A point can be rotated in 2D space if the center of rotation (or the pivot), is specified as well as the angle of rotation.
- Rotation Requires:-
  - Rotation axis
  - Rotation angle
  - Rotation point or pivot point (x_r, y_r)



2D Rotation

Rotation about a **pivot/arbitrary point**



- Pivot point is the point of rotation
- Pivot point need not necessarily be on the object

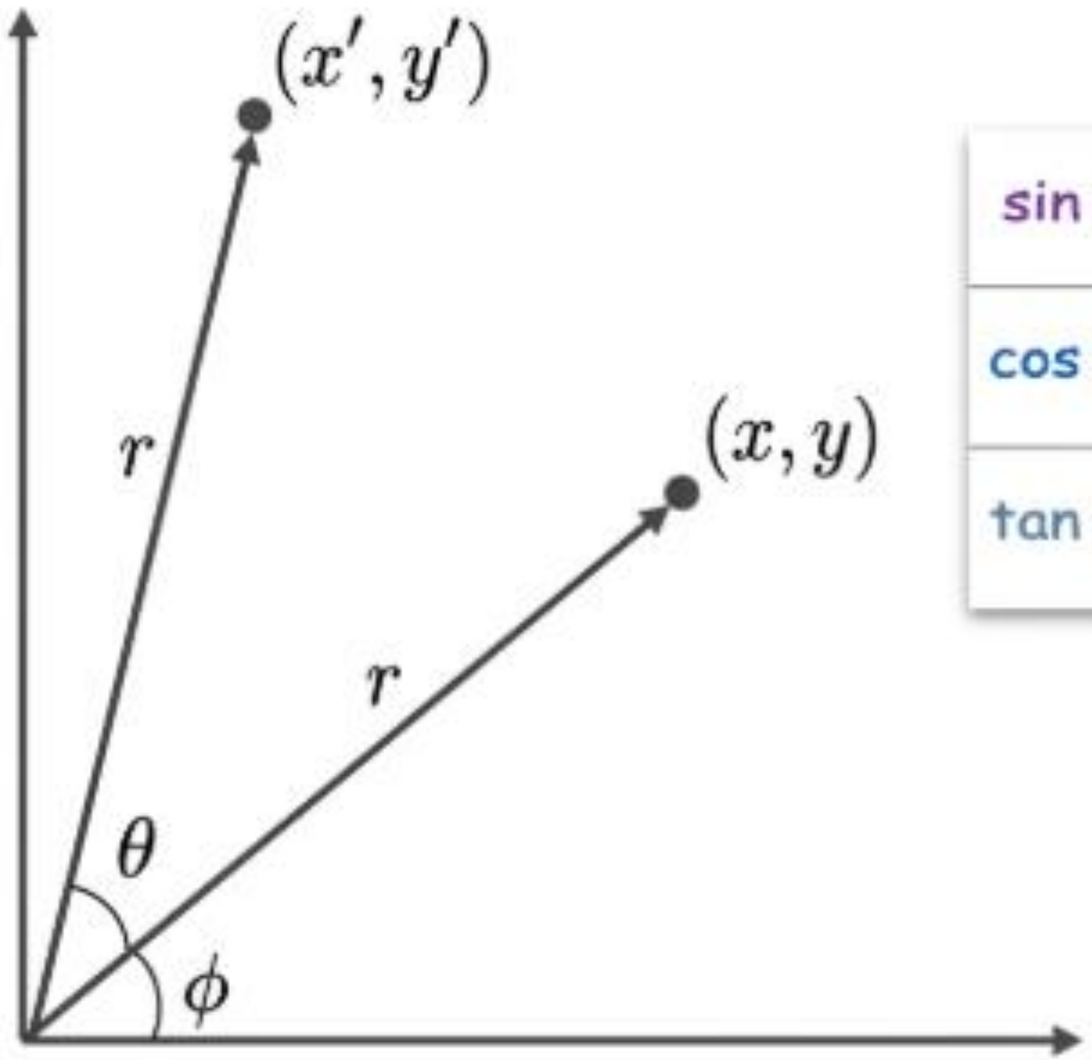
2D Rotation

- If θ is positive → counterclockwise rotation
- If θ is negative → clockwise rotation
- Rotations also move objects **without deformation**
- A line is rotated by applying the rotation formula to each of the endpoints and redrawing the line between the new end points
- A polygon is rotated by applying the rotation formula to each of the vertices and redrawing the polygon using new vertex coordinates
- **Remember:**

$$\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$$
$$\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$$

2-D Rotation

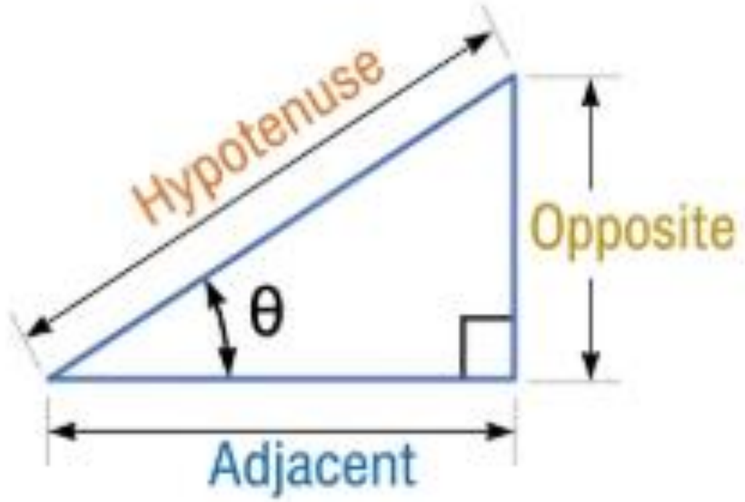
- Sine, **Cosine** and **Tangent** are the main functions used in **Trigonometry**



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



2D Rotation

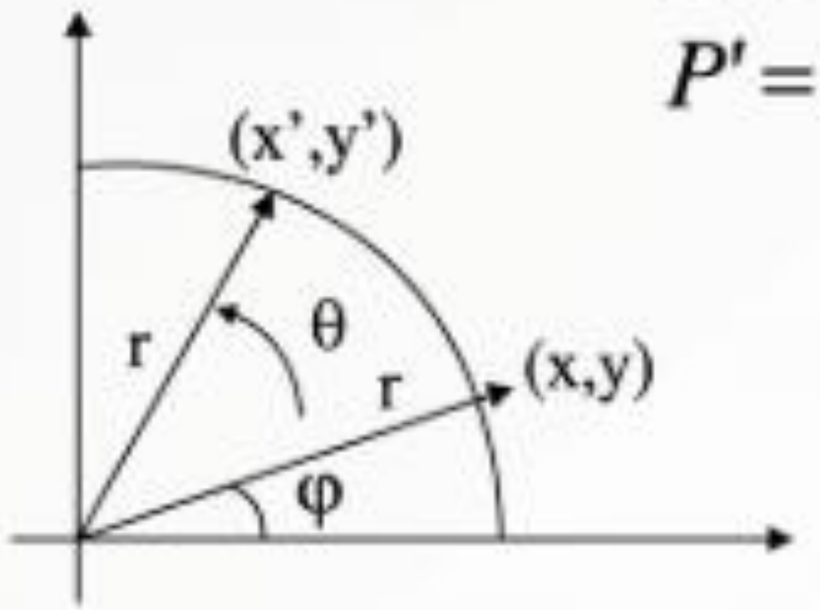
At first, suppose the pivot point is **at the origin**

$$P = (x, y)$$
$$R = (\theta)$$
$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$x' = r\cos(\varphi + \theta)$$
$$y' = r\sin(\varphi + \theta)$$

$$x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta$$
$$y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta$$
$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$P' = R \cdot P$$



2D Rotation

$$\text{Cos } \emptyset = \frac{x}{r} \quad , \quad x = r\cos \emptyset$$

$$\text{sin } \emptyset = \frac{y}{r} \quad , \quad y = r\sin \emptyset$$

$$x' = \text{r cos } (\theta + \emptyset) = r\cos\theta \cos\emptyset - r \sin\theta \sin\emptyset$$

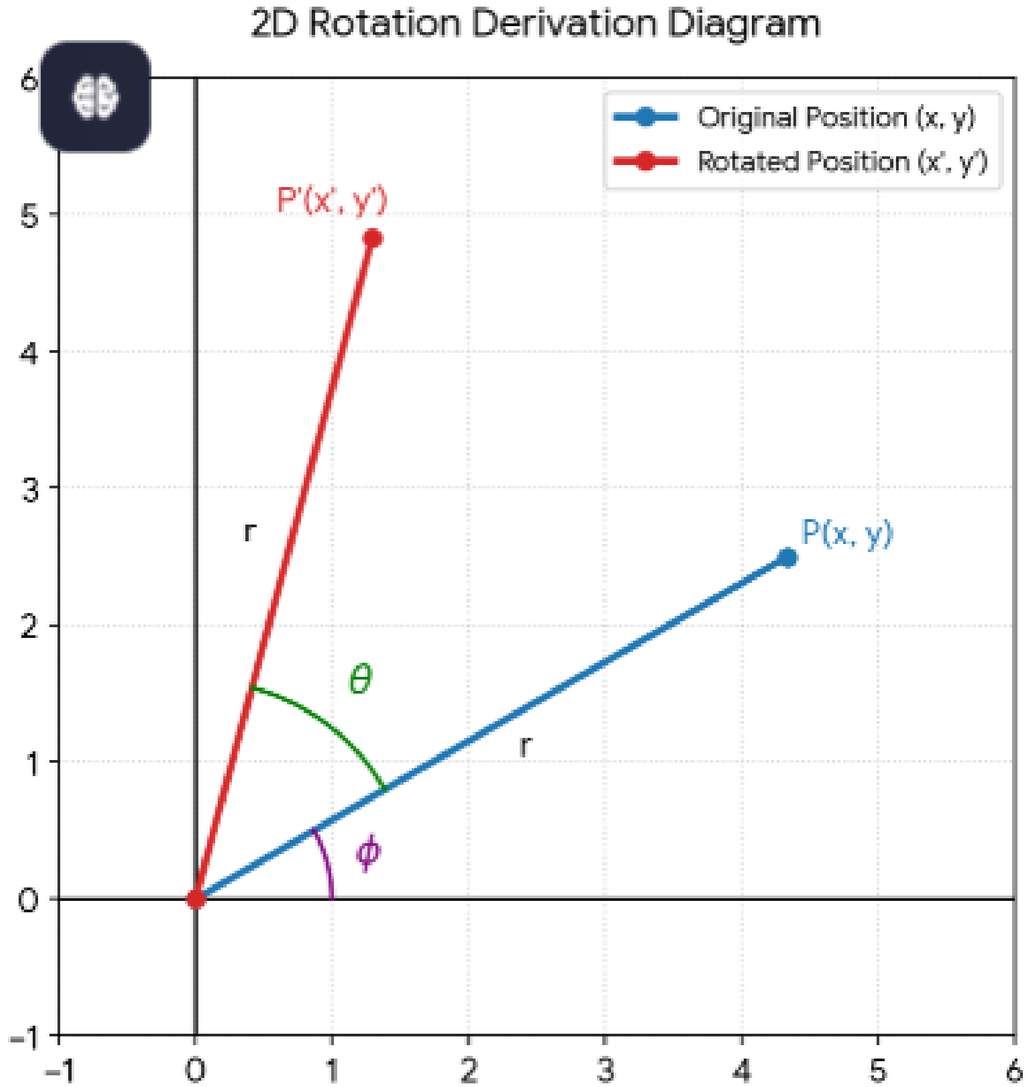
$$y' = \text{r sin } (\theta + \emptyset) = r\sin\theta \cos\emptyset + r \cos\theta \sin\emptyset$$

$$x' = \text{x cos } \theta - y \sin\theta$$

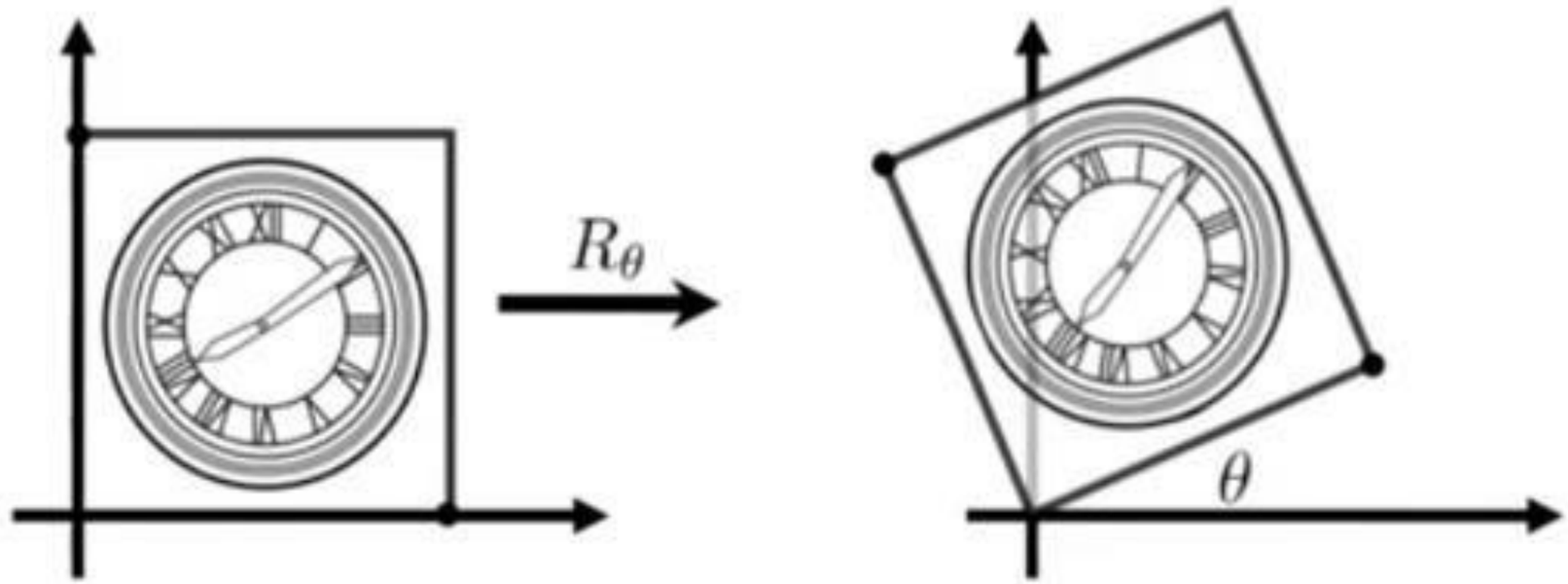
$$y' = \text{x sin } \theta + y \cos\theta$$

$$P' = R \cdot P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Rotation Matrix



$$x' = x\cos(\theta) - y\sin(\theta)$$
$$y' = x\sin(\theta) + y\cos(\theta)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

2D Rotation

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\vec{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\vec{\mathrm{R}}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}}_{\vec{\mathbf{p}}_1}$$

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathrm{R}(\theta)} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

Example

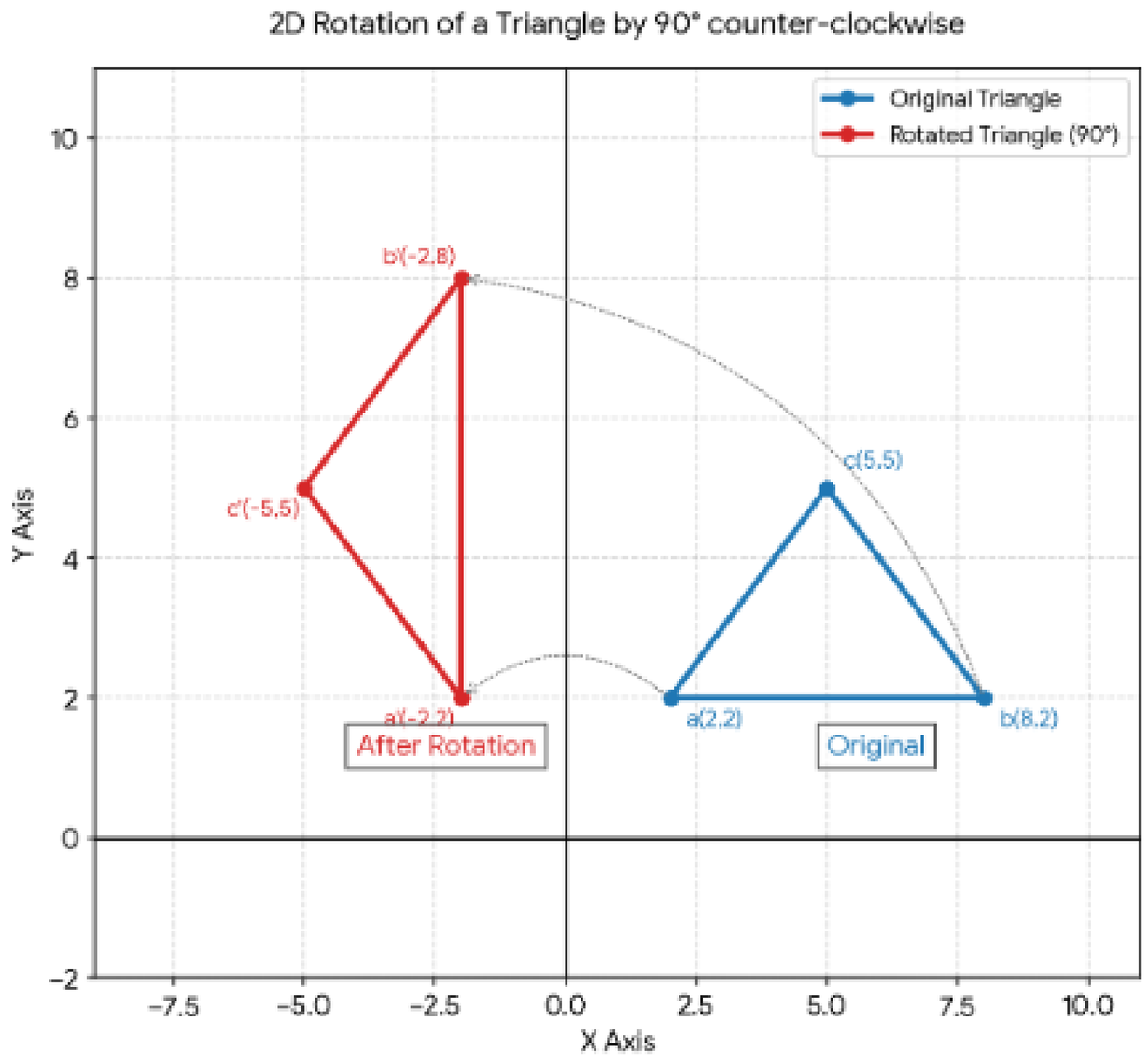
One triangle is given (2,2) (8,2) (5,5). Rotate the triangle 90°.

$$R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$C' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$



Example

Find the transformed point, p', caused by rotating P = (5, 1) about the origin through an angle of 90.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cdot \cos\theta - y \cdot \sin\theta \\ x \cdot \sin\theta + y \cdot \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot \cos 90^\circ - 1 \cdot \sin 90^\circ \\ 5 \cdot \sin 90^\circ + 1 \cdot \cos 90^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 0 - 1 \cdot 1 \\ 5 \cdot 1 + 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Example 2

A point (4, 3) is rotated in counter clock wise direction by the angle of 45. Find the rotation matrix R and the resultant points.

$$P' = R\,P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{bmatrix}_{2\times 1}$$

Example

Rotate a triangle A(0,0) , B(2,2) and C(4,2) about the origin by an angle of 45

$$A' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

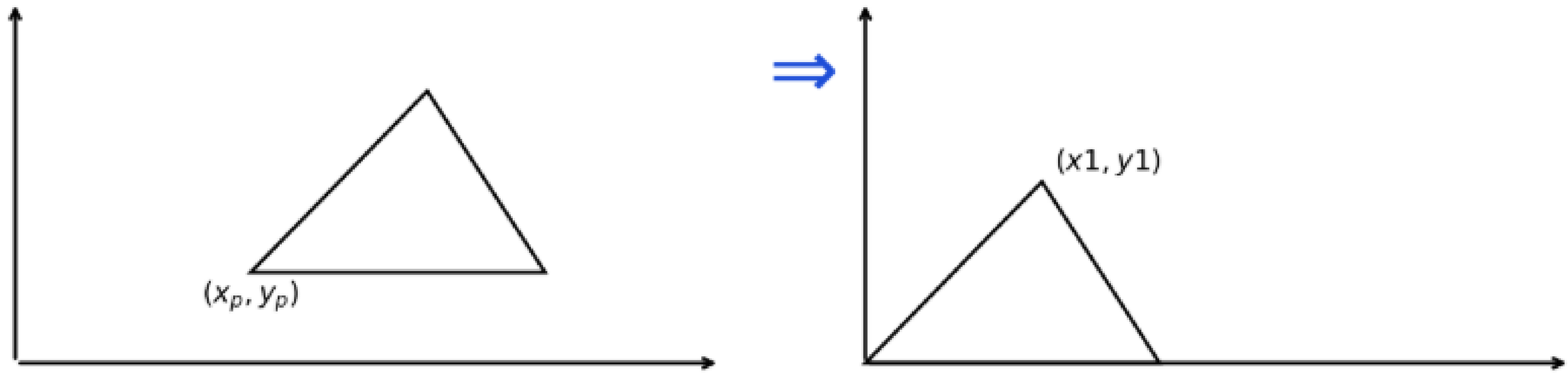
$$B' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4/\sqrt{2} \end{bmatrix}$$

$$C' = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{2} \\ 6/\sqrt{2} \end{bmatrix}$$

2D Rotation

Rotation about a pivot/arbitrary point

Step-1: Translate the pivot point to the origin

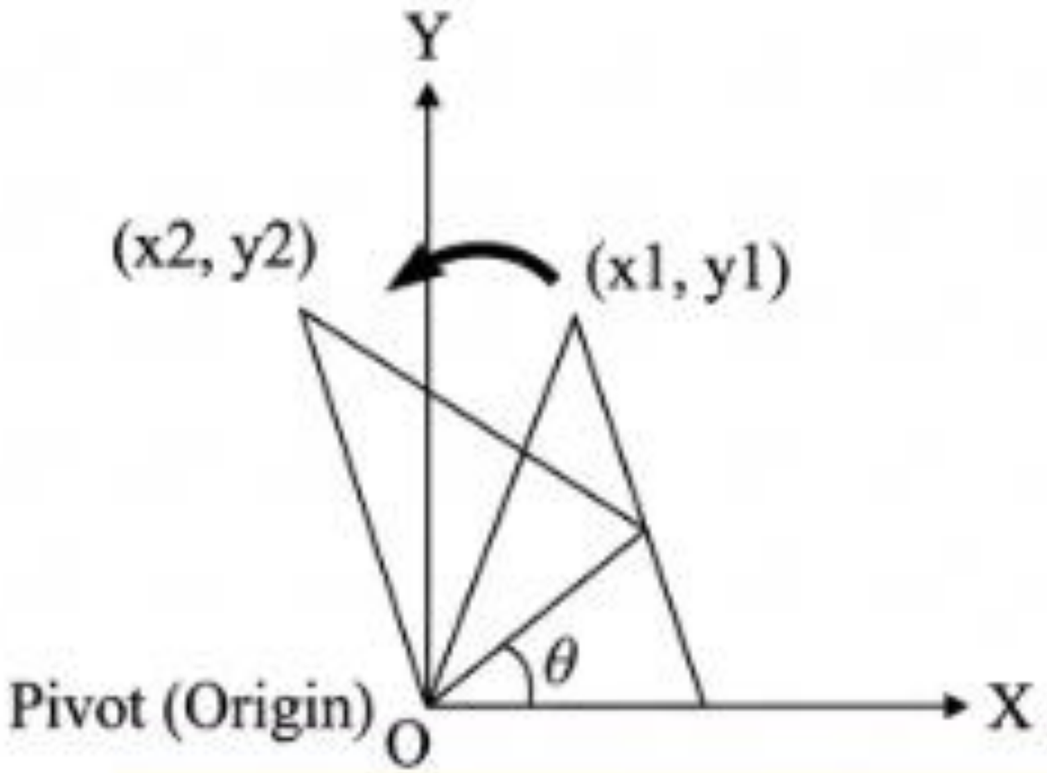


$$x1 = x - x_p$$
$$y1 = y - y_p$$

2D Rotation

Rotation about a **pivot/arbitrary point**

- **STEP-2:** Rotate about the origin



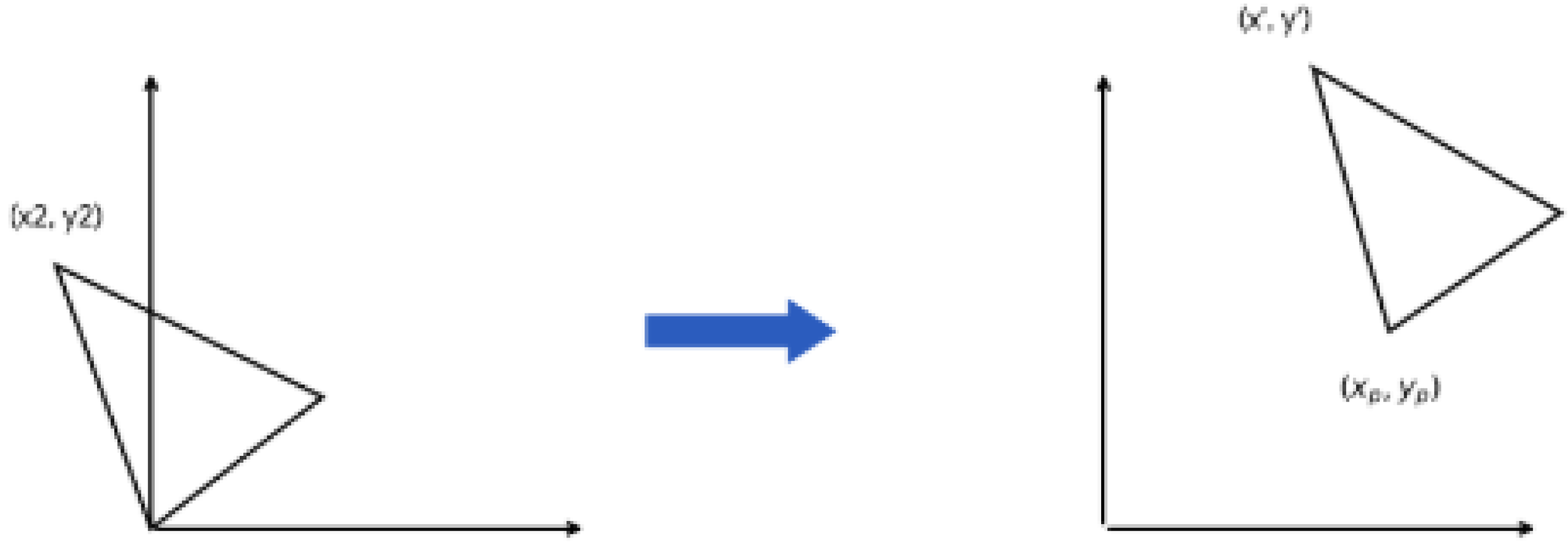
$$x2 = x1 \cos\theta - y1 \sin\theta$$

$$y2 = x1 \sin\theta + y1 \cos\theta$$

2D Rotation

Rotation about a pivot/arbitrary point

STEP – 3 : Translate the pivot point to original position



$$x' = x2 + x_p$$

$$y' = y2 + y_p$$

2D Rotation

$$\mathbf{M} = \mathrm{M}_3 \times \mathrm{M}_2 \times \mathrm{M}_1$$

Rotation about a **pivot/arbitrary point**

$$x' = (x - x_p)\cos\theta - (y - y_p)\sin\theta + x_p$$
$$y' = (x - x_p)\sin\theta + (y - y_p)\cos\theta + y_p$$

Inhomogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}}_{\hat{\mathbf{p}}_2} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{\hat{R}(\theta)} \underbrace{\begin{bmatrix} x_1 - x_0 \\ y_1 - y_0 \end{bmatrix}}_{(\hat{\mathbf{p}}_1 - \hat{\mathbf{p}}_0)} + \underbrace{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}}_{\hat{\mathbf{p}}_0}$$

Homogeneous coordinates

$$\underbrace{\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}}_{\mathbf{p}_2} = \underbrace{\begin{bmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T(x_0, y_0)^T} \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R(\theta)} \underbrace{\begin{bmatrix} 1 & 0 & -x_0 \\ 0 & 1 & -y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T(-x_0, -y_0)^T} \underbrace{\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}}_{\mathbf{p}_1}$$

$$= \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) & -x_0\cos(\theta) + y_0\sin(\theta) + x_0 \\ \sin(\theta) & \cos(\theta) & -x_0\sin(\theta) - y_0\cos(\theta) + y_0 \\ 0 & 0 & 1 \end{bmatrix}}_{T(x_0, y_0)^T R(\theta) T(-x_0, -y_0)^T} \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix}_{\mathbf{p}_1}$$

Example

Find the transformed point p1, caused by rotating p=(5,1) about the origin through an angle of 90.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \cdot \cos\theta - y \cdot \sin\theta \\ x \cdot \sin\theta + y \cdot \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot \cos 90^\circ - 1 \cdot \sin 90^\circ \\ 5 \cdot \sin 90^\circ + 1 \cdot \cos 90^\circ \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 0 - 1 \cdot 1 \\ 5 \cdot 1 + 1 \cdot 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

P1=(-1,5)

Example

A point (4,3) is rotated in counter clockwise direction by the angle of 45. Find the rotation matrix R and the resultant points.

$$P' = R \cdot P$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45^\circ & -\sin 45^\circ \\ \sin 45^\circ & \cos 45^\circ \end{bmatrix}_{2\times 2} \begin{bmatrix} 4 \\ 3 \end{bmatrix}_{2\times 1}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{\sqrt{2}} - \frac{3}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} + \frac{3}{\sqrt{2}} \end{bmatrix}_{2\times 1} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{7}{\sqrt{2}} \end{bmatrix}$$

Example

Rotate a triangle A(0,0) B(2,2) C(4,2) about the origin by an angle of 45.

Solution

Rotation matrix $R = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$A' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$B' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} - \frac{2}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{4}{\sqrt{2}} \end{bmatrix}$$

$$C' = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{\sqrt{2}} - \frac{2}{\sqrt{2}} \\ \frac{4}{\sqrt{2}} + \frac{2}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{2}{\sqrt{2}} \\ \frac{6}{\sqrt{2}} \end{bmatrix}$$

Example

One triangle is given (2,2), (8,2), (5,5). Rotate the triangle with an angle of 90

Rotation matrix

$$R = \begin{bmatrix} \cos 90^\circ & -\sin 90^\circ \\ \sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$B' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}$$

$$C' = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

