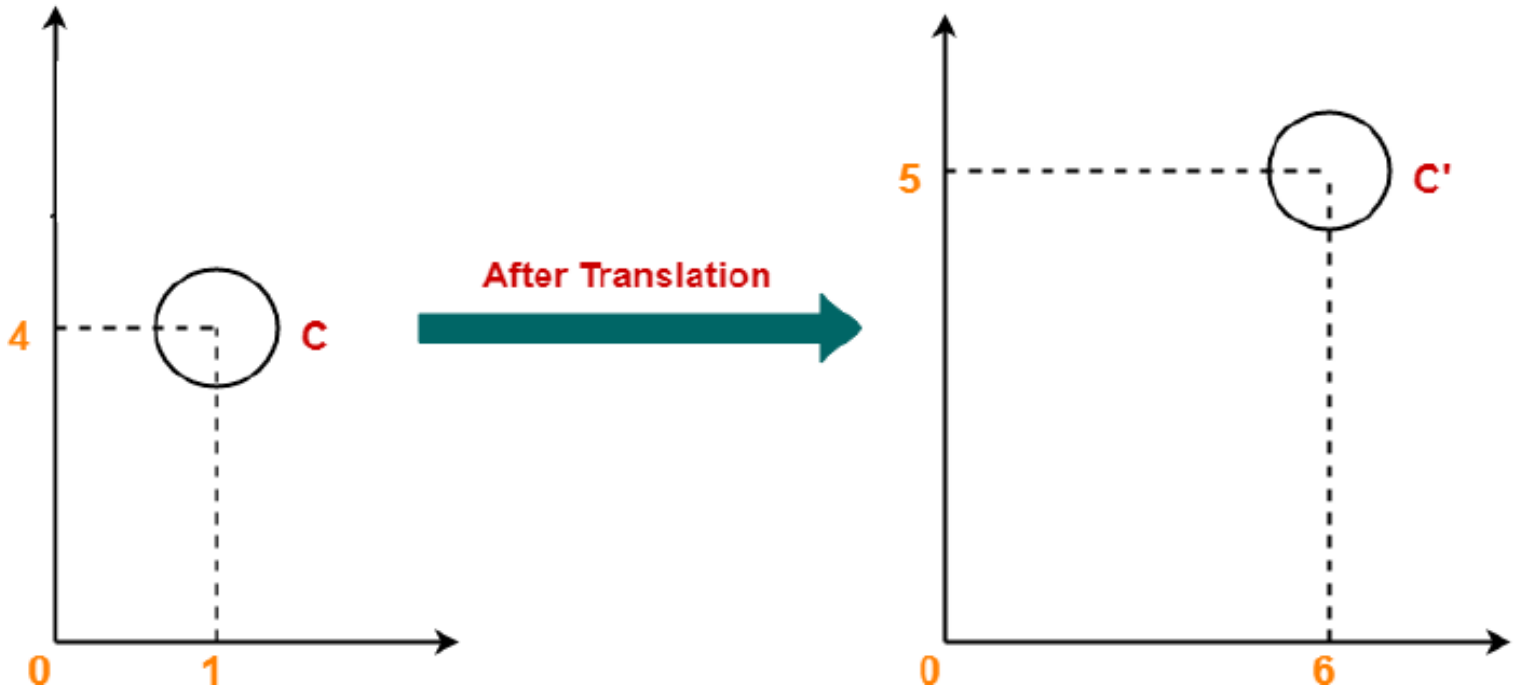
Lecture 3
transformation

Geometric Transformation Functions

- **Translations**
- **Scaling**
- **Rotation**

1. Translation



Translation Matrix

$$\begin{vmatrix} 1 & 0 & Tx \\ 0 & 1 & Ty \\ 0 & 0 & 1 \end{vmatrix}$$

Equation

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & Tx \\ 0 & 1 & Ty \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Example: (1)

A triangle has vertices a (2,2), b (10,2), and c (5,5). Translate the triangle using T_x=5 and T_y=6, and determine the new coordinates of the vertices.

Solution:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

T_x = 5 and T_y = 6 **a(2,2) → x = 2, y = 2**

$$\text{a'} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \mathbf{5} \\ 0 & 1 & \mathbf{6} \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{7} \\ \mathbf{8} \\ \mathbf{1} \end{bmatrix}$$

a' (7, 8) → **x ' = 7 and y ' = 8**

b' (15, 8) → **x ' = 15 and y ' = 8**

c' (10, 11) → **x ' = 10 and y ' = 11**

3D Translation

T(Tx, Ty) ⟶ T(Tx, Ty, Tz)

$$\begin{vmatrix} 1 & 0 & Tx \\ 0 & 1 & Ty \\ 0 & 0 & 1 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

Example: **Given a 3D object with coordinate points A(0, 3, 1), B(3, 3, 2), C(3, 0, 0), and D(0, 0, 0). Apply the translation with the distance 1 towards x-axis, 1 towards y-axis and 2 towards z- axis and obtain the new coordinates of the object.**

T(1, 1, 2)

Solution:

$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & Tx \\ 0 & 1 & 0 & Ty \\ 0 & 0 & 1 & Tz \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

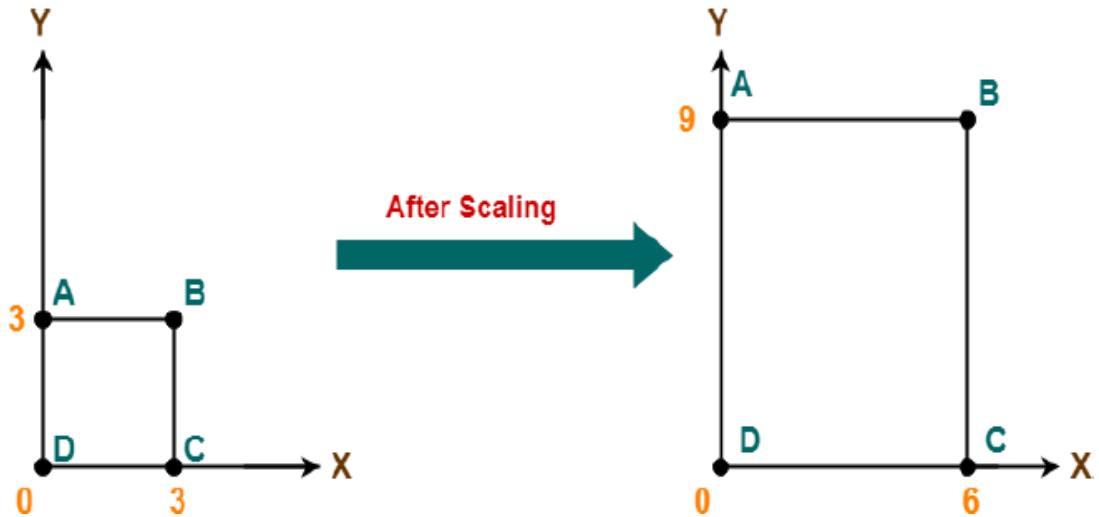
$$A' = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 3 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \times 0 + 1 \\ 1 \times 3 + 1 \\ 1 \times 1 + 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 4 \\ 3 \\ 1 \end{vmatrix} \quad A' (1, 4, 3)$$

$$B' = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 3 \\ 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \times 3 + 1 \\ 1 \times 3 + 1 \\ 1 \times 2 + 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 4 \\ 4 \\ 4 \\ 1 \end{vmatrix} \quad B' (4, 4, 4)$$

$$C' = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \times 3 + 1 \\ 1 \times 0 + 1 \\ 1 \times 0 + 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 4 \\ 1 \\ 2 \\ 1 \end{vmatrix} \quad C' (4, 1, 2)$$

$$D' = \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \times 0 + 1 \\ 1 \times 0 + 1 \\ 1 \times 0 + 2 \\ 1 \end{vmatrix} = \begin{vmatrix} 1 \\ 1 \\ 2 \\ 1 \end{vmatrix} \quad D' (1, 1, 2)$$

2. Scaling



Scaling Matrix

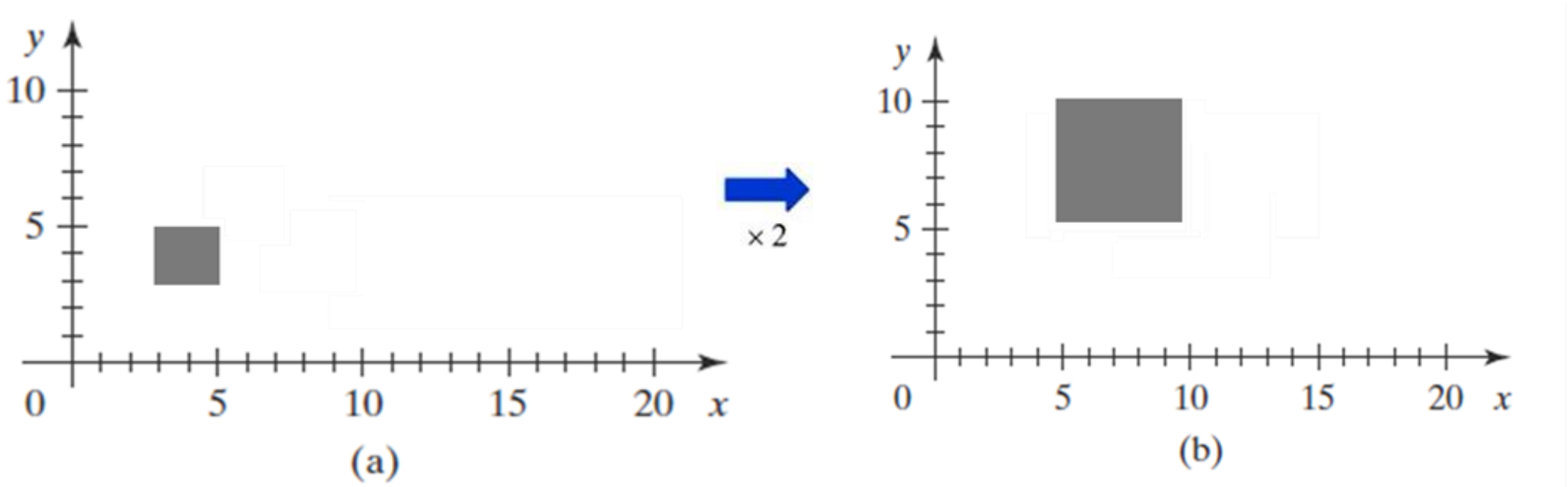
$$\begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Equation

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

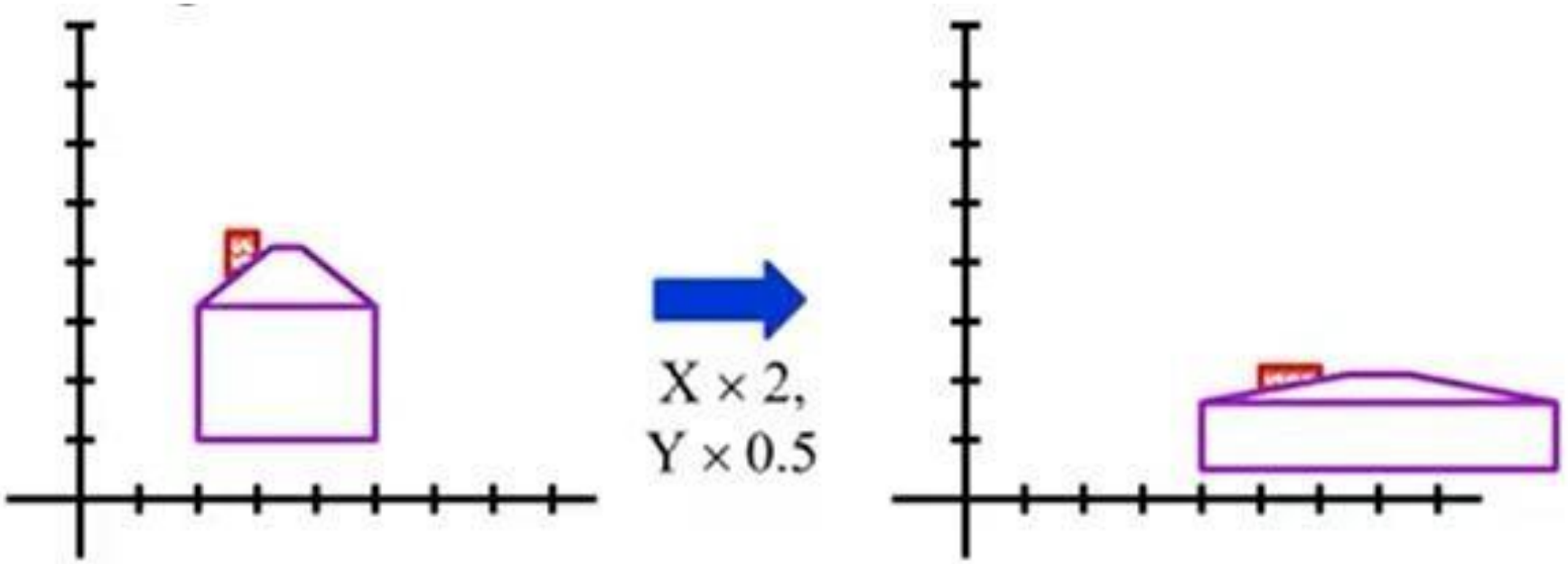
Uniform scaling:

Means the scalar is the same for all components



Non-uniform scaling:

Different scalars per component



Scaling

- **Multiplying object positions (x, y) by scaling factors s_x and s_y to produce the transformed coordinates (x', y'):**

$$x' = s_x \cdot x \qquad y' = s_y \cdot y$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix}}_{\text{scaling matrix}} \cdot \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling operation:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

3D Scaling

S(Sx, Sy) ⟶ S(Sx, Sy, Sz)

$$\begin{vmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{vmatrix} \longrightarrow \begin{vmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

Example: Given a 3D object with coordinate points A(0, 3, 3), B(3, 3, 6), C(3, 0, 1), and D(0, 0, 0). Apply the scaling parameter 2 towards x-axis, 3 towards y-axis and 3 towards z-axis and obtain the new coordinates of the object.

Solution:

$$\begin{vmatrix} x' \\ y' \\ z' \\ 1 \end{vmatrix} = \begin{vmatrix} Sx & 0 & 0 & 0 \\ 0 & Sy & 0 & 0 \\ 0 & 0 & Sz & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ z \\ 1 \end{vmatrix}$$

Scale shape S(2, 3, 3)

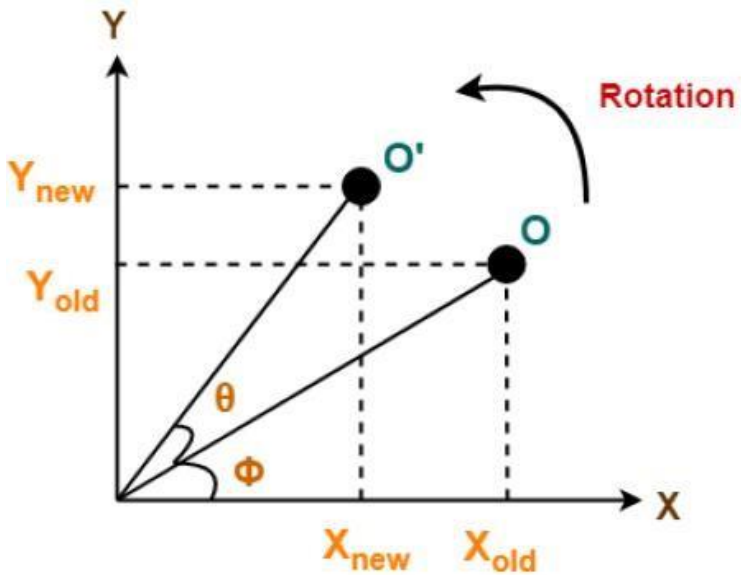
$$A' = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 3 \\ 3 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \times 0 \\ 3 \times 3 \\ 3 \times 3 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 9 \\ 9 \\ 1 \end{vmatrix} \quad A' (0, 9, 9)$$

$$B' = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 3 \\ 6 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \times 3 \\ 3 \times 3 \\ 3 \times 6 \\ 1 \end{vmatrix} = \begin{vmatrix} 6 \\ 9 \\ 18 \\ 1 \end{vmatrix} \quad B' (6, 9, 18)$$

$$C' = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 3 \\ 0 \\ 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \times 3 \\ 3 \times 0 \\ 3 \times 1 \\ 1 \end{vmatrix} = \begin{vmatrix} 6 \\ 0 \\ 3 \\ 1 \end{vmatrix} \quad C' (6, 0, 3)$$

$$D' = \begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 2 \times 0 \\ 3 \times 0 \\ 3 \times 0 \\ 1 \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \\ 0 \\ 1 \end{vmatrix} \quad D' (0, 0, 0)$$

3. Rotation



Rotation Matrix

$$\begin{vmatrix} \cos(\theta) & -sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Equation

$$\begin{vmatrix} x' \\ y' \\ 1 \end{vmatrix} = \begin{vmatrix} \cos(\theta) & -sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} \begin{vmatrix} x \\ y \\ 1 \end{vmatrix}$$

Example: A triangle has vertices a (2,2), b (8,2), and c (5,5). Rotate the triangle by 90 degrees, what are the new coordinates of the triangle after rotation.

Solution:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}(\theta) \cdot \mathbf{P}$$

θ = 90 degrees **cos (θ) = 0** **sin (θ) = 1**

$$\mathbf{a'} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{2} \\ 1 \end{bmatrix}$$

a' (-2, 2) → x ' = -2 and y ' = 2

b' (-2, 8) → x ' = -2 and y ' = 8

c' (-5, 5) → x ' = -5 and y ' = 5

Example: (2)

A triangle has vertices a (2,2), b (8,2), and c (5,5). Rotate the triangle by 90 degrees, what are the new coordinates of the triangle after rotation.

Solution:

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}(\theta) \cdot \mathbf{P}$$

θ = 90 degrees **cos (θ) = 0** **sin (θ) = 1**

$$\mathbf{a'} = \begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{-1} & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{2} \\ \mathbf{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{-2} \\ \mathbf{2} \\ 1 \end{bmatrix}$$

a' (-2, 2) → x ' = -2 and y ' = 2

b' (-2, 8) → x ' = -2 and y ' = 8

c' (-5, 5) → x ' = -5 and y ' = 5

Scaling around an arbitrary point



(a) Original Position of Object and Fixed Point

(b) Translate Object so that Fixed Point (x_f, y_f) is at Origin

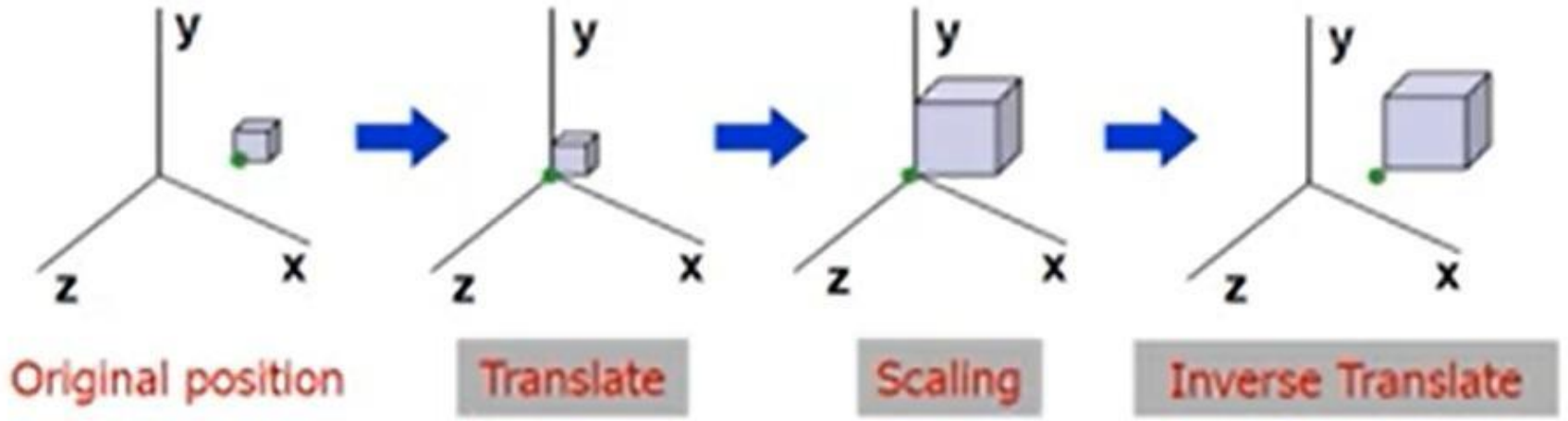
(c) Scale Object with Respect to Origin

(d) Translate Object so that the Fixed Point is Returned to Position (x_f, y_f)

Pipeline operations

- **Scaling with a selected fixed position**

  - Scaling with a Selected Fixed Position



$$T(x_f, y_f, z_f) \cdot S(s_x, s_y, s_z) \cdot T(-x_f, -y_f, -z_f) = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x_f \\ 0 & 1 & 0 & y_f \\ 0 & 0 & 1 & z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -x_f \\ 0 & 1 & 0 & -y_f \\ 0 & 0 & 1 & -z_f \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$